(INTRODUCTION TO) FUNCTION SPACES.

SOME QUESTIONS FOR SELF-CONTROL

DO YOU REMEMBER DEFINITIONS AND BASIC CONCEPTS?

- What is a *metric* and what is a *metric* space?
- What is a *norm* and what is a *normed* space?
- How to define the *convergent* sequences and *continuous* functions in a metric space?
- What is a *direct* ($f(V)$) and an *inverse* ($f^{-1}(V)$) images of a set V under the map f?
- How to define an ε-ball in a metric space?
- What is an *open* and a *closed* set? What is a *closure*?
- What metric spaces are called *separable*?
- What is a *Cauchy* sequence and what metric spaces are called *complete*?
- What is a *completion* of a metric space and what is $L^2([a, b])$?
- What function is called *uniformly* continuous and *Lipschitz* continuous?
- What is a *Banach* space?
- What norms on a vector space are called *equivalent*?
- What metric spaces are called *compact*? And what metric spaces are *totally bounded*?
- What is a *contraction* of a metric space X and what is a *fixed point* of a map $F : X \to X$?
- What is an *inner product* on a vector space and how to define the *norm* associated with the inner product?
- What is a *Hilbert* space?
- What is a *Cauchy-Schwartz* inequality?
- What is an *orthonormal/orthogonal* system in a Hilbert space?
- What is a *Fourier series* associated with an orthonormal system and what is a *Bessel* inequality?
- What orthonormal system is called *complete* and what is the *Parseval* equality?
- What are the *classical* Fourier series?

DO YOU KNOW MAIN RESULTS and NAMED THEOREMS?

- What can you say about different norms on $\mathbb{R}^n$?
- State the continuity criteriums for a function f on a metric space X via the convergent sequences and via the images of open sets.
- Formulate the basic properties of *open* and *closed* sets.
- State the Hausdorff criterium.
- State the Arzela theorem.
- State the Banach cotraction theorem.
- State the local solvability theorem for the systems of ODEs and explain how to reduce the ODE to a fixed point problem.
- When a Hilbert space does possess an orthonormal basis?
- State the Dirichlet theorem.
- When the *classical* Fourier series of a function f converges a) point-wise? b) in mean (in the L^2-metric)? c) uniformly?
- How to compute the coefficients for the classical Fourier series?

COULD YOU GIVE THE EXAMPLES (AND JUSTIFY THEM!)?

- Of a *complete* and a *non-complete* metric space;
- Of a *separable* and a *non-separable* metric space;
- Of two non-equivalent norms on $C[a,b]$;
- Of a *compact* and *non-compact*, but *complete* metric space;
- Of a contraction on a *non-complete* metric space which do not have any fixed points;
- Of a *complete* and *non-complete* orthonormal system in a Hilbert space;
- Of a function $f \in C^\infty[-\pi,\pi]$ such that the associated Fourier series does not converge *uniformly* to f;

ARE YOU ABLE TO PROVE (THAT)

- A ball $B_\varepsilon(x_0)$ is an *open* set?
- A union of any number of open sets is open and an intersection of any number of closed sets is closed?
- A metric/norm is a *continuous* function on the corresponding metric/normed space?
- A uniform limit of continuous functions is a continuous function?
- The spaces l_2 and l_∞ are complete?
- The unit closed ball in l_2 is *not* compact?
- The continuous image of a *compact* set is a compact set again ($f(K)$ is compact if K is compact and f is continuous)?
- The *compact* set is complete and totally bounded (without using the Hausdorff criterium!)?
- The Banach contraction principle?
- A scalar function $f : \mathbb{R} \to \mathbb{R}$ is (is not) a contraction using the Mean Value Theorem?
- The Cauchy-Schwartz inequality?
- Bessel inequality?

ARE YOU ABLE TO COMPUTE THE CLASSICAL FOURIER SERIES FOR

- $f(x) = \operatorname{sgn}(x)$?
- $f(x) = x$?
- $f(x) = e^x$?
- $f(x) = x^2$?

AND WRITE OUT THE PARSEVAL EQUALITY FOR ALL THAT EXAMPLES?