Classtest for "(Introduction to) Function Spaces" (MMath and BSc)

November 4, 2008

Problem 1. Let (X, d) be a metric space:

a) (1 point): Give the definitions of a *closure* $\bar{V}$ and of an interior point set $\mathrm{int}(V)$ of a set $V \subset X$.

b) (1 point): Prove that

$$\overline{V \cap W} \subset \overline{V} \cap \overline{W}$$

for any two subsets V and W of X.

c) (1 point): Give an example of two sets V and W on a real line $\mathbb{R}$ (with the standard metric) such that

$$\overline{V \cap W} \neq \overline{V} \cap \overline{W}.$$

Problem 2. Let (X, d) be a metric space:

a) (0.5 points): Give the definition of a Cauchy sequence in X. What does it mean when (X, d) is *complete*?

b) (0.5 points): What does it mean when (X, d) is compact?

c) (2 points): Let $V \subset X$ be a *compact* set in X and let $x_0 \in X$ be such that $x_0 \notin V$. Prove that

$$d(x_0, V) := \inf\{d(x_0, x)\,,\ x \in V\} > 0.$$

Problem 3.

a) (1 point): Let $X = \mathbb{R}^2$ ($x = (x_1, x_2) \in X$) and let

$$d(x, y) := |x_1 - y_1| + 20|x_2 - y_2|, \quad x, y \in \mathbb{R}^2.$$

Prove that (X, d) is a metric space.

b) (2 points): Let $F : \mathbb{R}^2 \to \mathbb{R}^2$ be defined by

$$F(x) := (\frac{1}{4}x_1 + 5x_2, \frac{1}{4}x_2), \quad x = (x_1, x_2) \in \mathbb{R}^2.$$

Prove that F is a *contraction* on a metric space (X, d) defined in question a).

c) (1 point): Is F a contraction on $\mathbb{R}^2$ with the standard Euclidean metric? Explain your answer.

Problem1.

a) The closure $\overline{V}$ of the set V consists of all points $x_0 \in X$ such that there exists a sequence $x_n \in V$ converging in X to x_0.

The interior point set $\mathrm{int}(V)$ of the set V consists of all points $x_0 \in V$ such that there exists an ε-ball $B_\varepsilon(x_0) \subset V$ for some positive ε.

b) Let $x_0 \in \overline{V \cap W}$. By definition, this means that there exists a sequence $x_n \in V \cap W$ such that $x_n \to x_0$ in X as $n \to \infty$. If $x_n \in V \cap W$ then $x_n \in V$ and $x_n \in W$. Thus, again by definition, $x_0 \in \overline{V}$ and $x_0 \in \overline{W}$ (recall that $x_n \to x_0$) and therefore $x_0 \in \overline{V} \cap \overline{W}$. Since x_0 is arbitrary, $\overline{V \cap W} \subset \overline{V} \cap \overline{W}$.

c) Of course, there are many such examples. The simplest one is $V = (0,1)$ and $W = (1,2)$. Then, $\overline{V \cap W} = \varnothing$, but $\overline{V} \cap \overline{W} = \{1\}$.

Another simple example: $V = \{\sqrt{2}\}$ and $W = \mathbb{Q}$. Then, $\overline{V \cap W} = V \cap W = \varnothing$, but $\overline{W} = \mathbb{R}$ and $\overline{V} \cap \overline{W} = \{\sqrt{2}\}$.

Problem 2.

a) A sequence $x_n \in X$ is a Cauchy sequence in X if, for any $\varepsilon > 0$ there exists $N = N(\varepsilon)$ such that

$$d(x_n, x_{n+m}) < \varepsilon, \quad \forall n \geq N, \quad \forall m \in \mathbb{N}.$$

A metric space (X, d) is complete is any Cauchy sequence has a limit in X.

b) A metric space (X, d) is compact if any sequence $x_n \in X$ has a convergent subsequence x_{n_k}.

c) Argue by contradiction. Assume that the assertion is wrong and

$$\inf\{d(x_0, x), \quad x \in V\} = 0.$$

Then, by the infinum definition, there exists a sequence $x_n \in V$ such that

$$d(x_0, x_n) \to 0.$$

Since V is compact, the sequence x_n contains a convergent subsequence $x_{n_k} \to y_0 \in V$. Since $x \to d(x_0, x)$ is a continuous function on X,

$$d(x_0, y_0) = d(x_0, \lim_{k\to\infty} x_{n_k}) = \lim_{k\to\infty} d(x_0, x_{n_k}) = 0.$$

Thus, $x_0 = y_0$. But, by construction, $y_0 \in V$ and by the assumption $x_0 \notin V$. Contradiction.

Problem 3.

a) Let us check the axioms of the metric: symmetry $d(x,y) = d(y,x)$ is obvious and the positivity is also immediate. So, we only need to verify the triangle inequality

$$d(x,y) = |x_1 - y_1| + 20|x_2 - y_2| = |(x_1 - z_1) + (z_1 - y_1)| + 20|(x_2 - z_2) + (z_2 - y_2)| \le$$
$$\le |x_1 - z_1| + |z_1 - y_1| + 20|x_2 - z_2| + 20|z_2 - y_2| = d(x,z) + d(y,z)$$

where we have used that

$$|X + Y| \le |X| + |Y|. \tag{1}$$

Thus, $d(x,y)$ is a metric on X.

2) Use again inequality (1):

$$d(F(x), F(y)) = |1/4x_1 + 5x_2 - 1/4y_1 - 5y_2| + 20|1/4x_2 - 1/4y_2| =$$
$$= |1/4(x_1 - y_1) + 5(x_2 - y_2)| + 5|x_2 - y_2| \le$$
$$\le 1/4|x_1 - y_1| + 5|x_2 - y_2| + 5|x_2 - y_2| \le 1/2(|x_1 - y_1| + 20|x_2 - y_2|) = 1/2d(x,y).$$

Thus, $F(x)$ is a contraction with the contraction factor 1/2.

c) F is not a contraction on $\mathbb{R}^2$ with the usual Euclidean metric. Indeed, let $x = (0,1)$ and $y = (0,1)$. Then

$$\|F(x) - F(y)\|^2 = \|(0,0) - (5, 1/4)\|^2 = 5^2 + 1/16 > 25.$$

Thus, $\|F(x) - F(y)\| > 5$, but $\|x - y\| = 1$ and F cannot be a contraction in the Euclidean metric.