Classtest for "(Introduction to) Function Spaces" (MMath and BSc)

November 6, 2009

Problem 1. Let (X, d) be a metric space:

a) (1 point): Give the definitions of a *closure* $\bar{V}$ and of an interior point set $\operatorname{int}(V)$ of a set $V \subset X$.

b) (1 point): Prove that

$$\operatorname{int}(V \cup W) \supset \operatorname{int} V \cup \operatorname{int} W$$

for any two subsets V and W of X.

c) (1 point): Give an example of two sets V and W in some metric space X such that

$$\operatorname{int}(V \cup W) \neq \operatorname{int} V \cup \operatorname{int} W.$$

Problem 2. Let (X, d) be a metric space:

a) (0.5 points): Define what does it mean that a sequence $x_n \in X$ is *convergent* to some $x_0 \in X$.

b) (0.5 points): Let $f : X \to Y$, where Y is another metric space. Define what does it mean that the function f is *continuous* at $x_0 \in X$.

c) (1 point): Let $X = l_1$ be a space of sumable sequences with the standard norm and let $\vec{e}_k$ be the k-th coordinate vector in it. Let $\vec{x} = (x_1, x_2, \cdots, x_n, \cdots) \in l_1$ be an arbitrary point and let $\vec{x}_k := \sum_{n=1}^{k} x_n \vec{e}_n \in l_1$. Prove that

$$\vec{x}_k \to \vec{x} \ \text{ in } \ l_1 \ \text{ as } k \to \infty.$$

d) (1 point): Let $X = l_\infty$ be the space of bounded sequences with the standard *sup*-norm, $\vec{x} \in l_\infty$ be arbitrary and $\vec{x}_k \in l_\infty$ be the same as in the previous question. Is it always true that

$$\vec{x}_k \to \vec{x} \ \text{ in } \ l_\infty \ \text{ as } k \to \infty?$$

Justify your answer.

Problem 3.

a) (1 point): Let (X, d) be a metric space. Define what does it mean that the metric space $(\tilde{X}, \tilde{d})$, $X \subset \tilde{X}$, is a *completion* of the metric space X.

b) (1 point): Give the definition of the Lebesgue space $L_2([0, 1])$ of square integrable functions.

c) (2 points): Let $f(x) := \frac{1}{\sqrt{x}}$. Does it belong a) to $L_2([0, 1])$; b) to $L_2([1, 2])$? Justify your answer

Problem 1.

a) The closure $\overline{V}$ of the set V consists of all points $x_0 \in X$ such that there exists a sequence $x_n \in V$ converging in X to x_0.

The interior point set $\mathrm{int}(V)$ of the set V consists of all points $x_0 \in V$ such that there exists an ε-ball $B_\varepsilon(x_0) \subset V$ for some positive ε.

b) Let $x_0 \in \mathrm{int}\, V \cup \mathrm{int}\, W$. By definition, $x_0 \in V$ or $x_0 \in W$ and there exists a ball $B_\varepsilon(x_0)$ such that $B_\varepsilon(x_0) \in V$ or $B_\varepsilon(x_0) \in W$. In both cases $B_\varepsilon(x_0) \in V \cup W$ and therefore $x_0 \in \mathrm{int}(V \cup W)$.

c) Of course, there are many such examples. The simplest one is $X = \mathbb{R}$ with the standard metric, $V = (0, 1]$ and $W = [1, 2)$. Then, $\mathrm{int}(V \cup W) = (0, 2)$, but $\mathrm{int}\, V \cup \mathrm{int}\, W = (0, 1) \cup (1, 2)$.

Problem 2.

a) A sequence $x_n \in X$ is convergent to $x_0 \in X$ if for every $\varepsilon > 0$ there exists exists $N = N(\varepsilon)$ such that

$$d(x_n, x_0) < \varepsilon, \quad \forall n \geq N.$$

b) A function $f : X \to Y$ is continuous at x_0 if for every $\varepsilon > 0$ there exists $\delta = \delta(\varepsilon) > 0$ such that

$$d(f(x), f(x_0)) \leq \varepsilon, \quad \forall x \in X \text{ such that } \; d(x, x_0) \leq \delta.$$

c) Note that $\vec{x} - \vec{x}_k = (0, 0, \cdots, 0, x_{k+1}, x_{k+2}, \cdots)$. By definition of the norm in l_1,

$$\|\vec{x} - \vec{x}_n\|_{l_1} = S_k := \sum_{n=k+1}^{\infty} |x_n|.$$

Since $\vec{x} \in l_1$, we have $\|\vec{x}\|_{l_1} := \sum_{n=1}^{\infty} |x_n| < \infty$ and therefore the "tails" $S_k \to 0$ as $k \to \infty$. Thus, $\vec{x}_k \to \vec{x}$ in l_1.

d) This statement is not always true in l_∞. Indeed, let $\vec{x} = (1, 1, 1, \cdots) \in l_\infty$. Then

$$\|\vec{x}_k - \vec{x}\|_{l_\infty} = 1$$

for all k and therefore we do not have the convergence.

Problem 3.

a) The space $\tilde{X}$ is a completion of the space $X \subset \tilde{X}$ if

1) $(\tilde{X}, \tilde{d})$ is complete;

2) X is dense in $\tilde{X}$;

3) $\tilde{d}(x, y) = d(x, y)$ for all $x, y \in X$.

b) The Lebesgue space $L_2([0,1])$ is a completion of the space $C([0,1])$ with respect to the following norm:

$$\|f\|_{L_2} := \sqrt{\int_0^1 |f(x)|^2\,dx}.$$

c) The function $f \notin L_2(0,1)$ since the Riemann integral $\int_0^1 |f(x)|^2\,dx = \infty$. Indeed, since $x = 0$ is a singular point of f,

$$\int_0^1 |f(x)|^2\,dx = \lim_{\varepsilon\to 0+} \int_\varepsilon^1 \frac{dx}{x} = \lim_{\varepsilon\to 0} \ln\frac{1}{\varepsilon} = \infty.$$

Obviously, $f \in L_2([1,2])$ since $f \in C[1,2]$.