Classtest for "(Introduction to) Function Spaces" (MMath and BSc)

November 2, 2010

Problem 1. Let (X, d) be a metric space:

a) (0.5 point): Give the definitions of an *open* and of a *closed* set $V \subset X$.

b) (1.5 point): Let V and W be two *closed* sets of X. Prove by first principles that the union $V \cup W$ is closed.

c) (1 point): Let V and W be two *open* sets of X. Can their *intersection* $V \cap W$ be closed? Justify your answer.

Problem 2. Let X be a vector space:

a) (0.5 points): Define what does it mean that two norms $\|\cdot\|_1$ and $\|\cdot\|_2$ are *equivalent*.

b) (1 point): Let $\|\cdot\|_1$ and $\|\cdot\|_2$ be two norms on X and let a sequence $f_n \in X$ be such that

$$\|f_n\|_1 = n^2, \quad \|f_n\|_2 = 2n + 1.$$

Can these two norms be equivalent? Justify your answer.

c) (2 points): Let $X := C_0^1[0, 1]$ be a space of continuously differentiable functions on $[0, 1]$ such that $f(0) = 0$ and let

$$\|f\|_1 := \sup_{x\in[0,1]} |f'(x)|, \quad \|f\|_2 := \sup_{x\in[0,1]} |f(x)| + \sup_{x\in[0,1]} |f'(x)|$$

Prove that these two norms are equivalent.

Problem 3.

a) (1 point): Let (X, d) be a metric space. Define what does it mean that $x_n \in X$ is a *Cauchy sequence* and what does it mean that X is *complete*.

b) (1 point): Give the definition of the Lebesgue space $L_1([0, 1])$ of integrable functions.

c) (1.5 points): Let $f(x) := \frac{1}{x}$. Does it belong a) to $L_1([0, 1])$; b) to $L_1([1, 2])$? Justify your answer

Solutions

Problem 1: a) A set $V \subset X$ is open if for every $x \in V$ there is $\varepsilon > 0$ such that $B_\varepsilon(x) \subset V$. A set V is closed if for every sequence $x_n \in V$ such that $x_n \to x_0 \in X$, we have $x \in V$.

b) Let V and W be two closed sets and let $x_n \in V \cup W$ be a sequence such that $x_n \to x_0 \in X$. We need to prove that $x_0 \in V \cup W$. Since $x_n \in V \cup W$, $x_n \in V$ or $x_n \in W$. Then, at least one of sets V or W contains infinitely many terms of x_n. Let it be set V (the case of the set W is analogous). That means: there is a subsequence $x_{n_k} \in V$ such that $x_{n_k} \to x_0$. Since V is closed, $x_0 \in V$ and, therefore, $x_0 \in V \cup W$.

c) We know that intersection of two open sets is always open, but there are sets which are open and closed simultaneously! For instance, the empty set is always open and closed (if two open sets have empty intersection their intersection is closed). More interesting examples can be constructed using, e.g., space of totally disjoint points.

Problem 2: a) Two norms are equivalent if there are two positive numbers l and L such that

$$l\|x\|_1 \leq \|x\|_2 \leq \|x\|_1$$

for all $x \in X$.

b) These norms are not equivalent. Indeed, the inequality $l\|f_n\|_1 \leq \|f_n\|_2$ does not hold for all n.

c) Obviously, $\|f\|_1 \leq \|f\|_2$, so we only need to check the opposite inequality. Namely, it is enough to prove that

$$\sup_{x\in[0,1]} |f(x)| \leq \sup_{x\in[0,1]} |f'(x)|$$

To this end, we need to use that $f(0) = 0$ and therefore $f(x) = \int_0^x f'(s)\, ds$ and

$$|f(x)| \leq \int_0^1 |f'(s)|\, ds \leq \sup_{x\in[0,1]} |f'(x)| \int_0^1 ds = \sup_{x\in[0,1]} |f'(x)|.$$

Problem 3: a) A sequence x_n is a Cauchy sequence if, for every $\varepsilon > 0$ there exists $N = N(\varepsilon)$ such that

$$d(x_n, x_{n+m}) < \varepsilon$$

for all $n > N(\varepsilon)$ and all m. A space is complete if any Cauchy sequence is convergent.

b) A space $L_1(0,1)$ is a completion of the space of continuous functions $C[0,1]$ with respect to the norm $\|f\|_{L_1} := \int_0^1 |f(x)|\, dx$.

c) The function f belongs to $L_1(1,2)$ since it is continuous on that interval. Now, consider the case of $L_1[0,1]$. Here, the function f has a singularity at $x = 0$ and we need to use the criterium with the improper Riemann integral. Namely,

$$\int_0^1 |f(x)|\, dx = \lim_{\varepsilon\to 0} \int_\varepsilon^1 \frac{dx}{x} = \lim_{\varepsilon\to 0} \log\frac{1}{\varepsilon} = \infty.$$

Therefore, $f \notin L_1(0,1)$.