Class Test, Introduction to Function Spaces, 2006-07, MS310 (B.Sc.).

Problem 1:

Let $(f_n)_{n\in\mathbb{N}}$ be a sequence of functions in $C([0,1])$ defined by

$$f_n(x) = \frac{1}{n}x^n.$$

a) What is the pointwise limit of $(f_n)_{n\in\mathbb{N}}$? [2]

b) What is the uniform limit function f? [2]

c) Does the sequence of derivatives $(f_n')_{n\in\mathbb{N}}$ converge pointwise and/or uniformly? Justify your answer. [3]

d) Let $(g_n)_{n\in\mathbb{N}}$ be a sequence in $C([0,1])$ such that for each $n \in \mathbb{N}$:

$$g_n(0) = 0 \text{ and } g_n \text{ is differentiable with continuous derivative } g_n'.$$

Assume that $g_n' \to 0$ uniformly (here 0 indicates the 0-function).

Show that also $g_n \to 0$ uniformly. (**Hint:** Use the Mean Value Theorem.) [3]

Problem 2:

a) State the parallelogram law and explain what it is used for. [2]

b) Let $(X, \langle\ ,\ \rangle)$ be an inner product space, and $\|x\| = \sqrt{\langle x, x\rangle}$ be the corresponding norm. Prove that $\|\ \|$ satisfies the parallelogram law. [2]

c) Let $X = L^2([0,1], \mathbb{C})$ the space of L^2-functions $f : [0,1] \to \mathbb{C}$ with standard complex inner product. For which $a \in \mathbb{C}$ is the function ix perpendicular to $x + a$? [3]

d) State the Cauchy-Schwarz inequality for the Hilbert space from part c), and hence prove that

$$\int_0^1 x^4 e^{2x}\, dx \leq \frac{1}{6}\sqrt{e^4 - 1}.$$

[3]

Please turn over the page.

Problem 3:

a) What is a contraction on a metric space (X, d)? [2]

b) Let $T : L^1([0,1]) \to L^1([0,1])$ be defined as

$$(Tf)(x) = \int_0^x \frac{\sin t + f(t)}{2} \, dt.$$

Show that $|Tf(x) - Tg(x)| \leq \frac{1}{2}\|f - g\|_1$, and hence that T is a contraction. [3]

c) Let f_* be the fixed point of T. Which differential equation is solved by f_* (with which initial condition)?
Next show that if $f_0(t) \equiv 0$ and $f_n = Tf_{n-1}$ for $n \geq 1$, then $\|f_n - f_*\|_1 \leq 2^{-n}\|f_*\|_1$. [3]

d) Find all contractions on (X, d) if X is some space with the discrete metric d. [2]