Sobolev and spectral inequalities for orthonormal systems in the theory of attractors I
Berezin–Li&Yau inequalities

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Berezin inequality (1972)

Let $\Omega \subset \mathbb{R}^d$ with $|\Omega| := \mathrm{vol}(\Omega) < \infty$. The eigenvalue problem for the Dirichlet Laplacian

$$-\Delta\varphi_n = \lambda_n\varphi_n, \quad \varphi_n|_{\partial(\Omega)} = 0.$$

The system $\{\varphi_n\}_{n=1}^{\infty} \in H_0^1(\Omega)$ is orthonormal in $L_2(\Omega)$.
We extend the φ_n's by zero to the whole of $\mathbb{R}^d$ and use the Fourier transform. For a $\Lambda > 0$ using $\|\varphi_n\|^2 = 1$, $\|\nabla\varphi_n\|^2 = \lambda_n$ and the Plancherel identity

$$\sum_n (\Lambda-\lambda_n)_+ = \sum_{\lambda_n<\Lambda} \int_\Omega (\Lambda|\varphi_n|^2 - |\nabla\varphi_n|^2)dx = \sum_{\lambda_n<\Lambda} \int_{\mathbb{R}^d} (\Lambda-|\xi|^2)|\widehat{\varphi}_n|^2 d\xi$$
$$\leq \sum_n \int_{\mathbb{R}^d} (\Lambda - |\xi|^2)_+ |\widehat{\varphi}_n|^2 d\xi = \int_{\mathbb{R}^d} (\Lambda - |\xi|^2)_+ \sum_n |\widehat{\varphi}_n|^2 d\xi$$

We consider the sum $\sum_{n=1}^\infty |\varphi_n(\xi)|^2$. We interpret it as the sum of the squared Fourier coefficients of the expansion of the exponential $(2\pi)^{-d/2}e^{i\xi x}_{x\in\Omega}$ (where ξ is a parameter) with respect to the orthonormal system $\{\varphi_n\}_{n=1}^\infty$. By Parseval's identity, we have pointwise for $\xi \in \mathbb{R}^d$

$$\sum_{n=1}^\infty |\varphi_n(\xi)|^2 = \sum_{n=1}^\infty |(\varphi_n(\cdot), (2\pi)^{-d/2}e^{i\xi\cdot})_{L_2(\Omega)}|^2$$
$$= \|(2\pi)^{-d/2}e^{i\xi\cdot}\|^2_{L_2(\Omega)} = (2\pi)^{-d}|\Omega|.$$

Since

$$(2\pi)^{-d}\int_{\mathbb{R}^d}(\Lambda - |\xi|^2)_+ d\xi = \frac{1}{(4\pi)^{d/2}\Gamma(2+d/2)}\Lambda^{1+d/2} = L^{\mathrm{cl}}_{1,d}\Lambda^{1+d/2},$$

where

$$L^{\mathrm{cl}}_{\sigma,d} := (2\pi)^{-d}\int_{\mathbb{R}^d}(1-|\xi|^2)^\sigma_+ d\xi = \frac{\Gamma(\sigma+1)}{(4\pi)^{d/2}\Gamma(\sigma+d/2+1)},$$

we finally obtain

Theorem

$$\sum_n (\Lambda - \lambda_n)_+ \le L^{\mathrm{cl}}_{1,d} \Lambda^{1+d/2} \cdot |\Omega|.$$

Remark

The constant on the right-hand side is sharp.

Remark

The above proof shows that

$$\sum_n (\Lambda - \nu_n)_+ \le L^{\mathrm{cl}}_{1,d} \Lambda^{1+d/2} \cdot |\Omega|,$$

where $\nu_n := \|\nabla\psi_n\|^2$, *and* $\{\psi_n\}_{n=1}^{\infty} \in H^1_0(\Omega)$ *is an arbitrary system that is orthonormal in* $l_2(\Omega)$.

Li–Yau lower bound 1983

Theorem

Let $\{\varphi_n\}_{n=1}^{\infty} \in H_0^1(\Omega)$ *be an arbitrary system that is orthonormal in* $L_2(\Omega)$. *Then*

$$\sum_{n=1}^{N} \|\nabla \varphi_n\|^2 \geq \frac{d}{d+2} \left(\frac{(2\pi)^d}{\omega_d |\Omega|} \right)^{2/d} \cdot N^{1+2/d}$$

Proof. Using the previous notation we have for $F_N(\xi) := \sum_{n=1}^N |\widehat{\varphi}_n(\xi)|^2$

$$\int_{\mathbb{R}^d} F_N(\xi) = \sum_{n=1}^N \int_{\mathbb{R}^d} |\widehat{\varphi}_n(\xi)|^2 d\xi = \sum_{n=1}^N \int_\Omega |\varphi_n(x)|^2 dx = N,$$

$$\int_{\mathbb{R}^d} |\xi|^2 F_N(\xi) d\xi = \sum_{n=1}^N \int_{\mathbb{R}^d} |\widehat{\varphi}(\xi)|^2 d\xi = \sum_{n=1}^N \int_\Omega |\nabla \varphi_n(x)|^2 dx = \sum_{n=1}^N \|\nabla \varphi_n\|^2.$$

In addition, Bessel's inequality gives that

$$F_N(\xi) = \sum_{n=1}^N |(\varphi_n(\cdot), (2\pi)^{-d/2} e^{i\xi\cdot})_{L_2(\Omega)}|^2$$
$$\le \|(2\pi)^{-d/2} e^{i\xi\cdot}\|^2_{L_2(\Omega)} = (2\pi)^{-d}|\Omega|.$$

We now set

$$\Lambda := \left(\frac{(2\pi)^d}{\omega_d|\Omega|}\right)^{2/d} \cdot N^{2/d},$$

so that

$$N = (2\pi)^{-d}|\Omega| \int_{|\xi|^2<\Lambda} d\xi = \int_{\mathbb{R}^d} F_N(\xi),$$

and write

$$\int_{\mathbb{R}^d} |\xi|^2 F_N(\xi) d\xi = (2\pi)^{-d}|\Omega| \int_{|\xi|^2<\Lambda} |\xi|^2 d\xi + R,$$

where

$$R = \int_{|\xi|^2<\Lambda} |\xi|^2 \bigl(F_N(\xi) - (2\pi)^{-d}|\Omega|\bigr) d\xi + \int_{|\xi|^2>\Lambda} |\xi|^2 F_N(\xi) d\xi$$

$$= \int_{|\xi|^2<\Lambda} (|\xi|^2 - \Lambda)\bigl(F_N(\xi) - (2\pi)^{-d}|\Omega|\bigr) d\xi + \int_{|\xi|^2>\Lambda} |(\xi|^2 - \Lambda) F_N(\xi) d\xi \geq 0.$$

This finally gives that

$$\sum_{n=1}^{N} \|\nabla \varphi_n\|^2 = \int_{\mathbb{R}} |\xi|^2 F_N(\xi) d\xi$$

$$\geq (2\pi)^{-d} |\Omega| \int_{|\xi|^2 < \Lambda} |\xi|^2 d\xi = \frac{d}{d+2} \left(\frac{(2\pi)^d}{\omega_d |\Omega|} \right)^{2/d} \cdot N^{1+2/d}.$$

The proof is complete.

Remark

The constant is sharp, since if we take for the φ_n's the orthonormal eigenfunctions, then

$$\|\nabla \varphi_n\|^2 = \lambda_n \sim \left(\frac{(2\pi)^d}{\omega_d |\Omega|} \right)^{2/d} \cdot n^{2/d} \text{ as } n \to \infty.$$

Legendre duality

The Berezin upper bound for the Riesz means is equivalent to the Li–Yau lower bound and the equivalence is realized by the Legendre transform.
We recall that given a convex increasing function $f(\lambda)$ on $\mathbb{R}_+$ its Legendre transform is given by

$$\widetilde{f}(\nu) := \sup_{\lambda>0}(\lambda\nu - f(\lambda)).$$

If $f(\lambda) \le g(\lambda)$, then clearly

$$\widetilde{f}(\nu) \ge \widetilde{g}(\nu).$$

A straight forward calculation shows that the Legendre transform of a piecewise linear function

$$f(\lambda) = \sum_n (\lambda - b_n)_+, \quad b_n > 0, \quad b_1 \le b_2 \le \cdots \to \infty$$

is a piecewise linear function

$$\widetilde{f}(\nu) = \sum_{n=1}^{[\nu]} b_n + \{\nu\} b_{[\nu]+1}, \ \nu = [\nu] + \{\nu\}, \ 0 \le \{\nu\} < 1,$$

and other way round.
Applying the Legendre transform to both sides of the Berezin inequality and setting $\nu = N$ we obtain the Li–Yau inequality. Hence, the Berezin–Li&Yau inequality.

Stokes operator

Theorem

Let a system of vector functions $\{u_n\}_{n=1}^N \in \boldsymbol{H}_0^1(\Omega)$ *be orthonormal*

$$\int_\Omega u_k(x)\cdot u_l(x)dx = \delta_{k\,l}$$

and further let $\operatorname{div} u_n = 0$, $n = 1, \dots, n$. *Then*

$$\sum_{n=1}^N \|\nabla u_n\|^2 \geq \frac{d}{d+2}\left(\frac{(2\pi)^d}{(d-1)\cdot\omega_d|\Omega|}\right)^{2/d}\cdot N^{1+2/d}.$$

As before we set $F_N(\xi) = \sum_{n=1}^N |\widehat{u}_n(\xi)|^2$ and as before by Plancherel

$$\int_{\mathbb{R}^d} F_N(\xi)d\xi = N, \qquad \int_{\mathbb{R}^d} |\xi|^2 F_N(\xi)d\xi = \sum_{n=1}^N \|\nabla u_n\|^2.$$

How about the pointwise estimate for $F_N(\xi)$?

Lemma

$$F_N(\xi) \le (d-1)(2\pi)^{-d}|\Omega|.$$

The proof is a series of reductions.
We already now that for an arbitrary scalar orthonormal system

$$\sum_{n=1}^N |\widehat{\varphi}_n(\xi)|^2 \le (2\pi)^{-d}|\Omega|.$$

We now show that this inequality holds for a suborthonormal system.

Definition

A system $\{\varphi_n\}_{n=1}^N$ is called suborthonormal if

$$\sum_{i,j=1}^N \xi_i \bar{\xi}_j (\varphi_i, \varphi_j) \le \sum_{j=1}^N |\xi_j|^2 \quad \forall\, \xi \in \mathbb{C}^N.$$

If a system $\{\varphi_n\}_{n=1}^N$ is suborthonormal, then

$$\sum_{n=1}^N |\widehat{\varphi}_n(\xi)|^2 \le (2\pi)^{-d} |\Omega|.$$

In fact, by suborthonormality

$$
\begin{aligned}
0 \le \\
&\int\left((2\pi)^{-d/2}e^{-i\xi x}-\sum_{k=1}^{N}\widehat{\varphi}_k(\xi)\varphi_k(x)\right)\overline{\left((2\pi)^{-d/2}e^{-i\xi x}-\sum_{l=1}^{N}\widehat{\varphi}_l(\xi)\varphi_k(x)\right)}dx \\
&= (2\pi)^{-d}|\Omega| - 2\sum_{k=1}^{N}|\varphi_k(\xi)|^2 + \sum_{k,l=1}^{N}\widehat{\varphi}_k(\xi)\bar{\widehat{\varphi}}_l(\xi)(\varphi_k,\varphi_l) \\
&\le (2\pi)^{-d}|\Omega| - \sum_{n=1}^{N}|\widehat{\varphi}_n(\xi)|^2.
\end{aligned}
$$

Next, it is very easy to see that if a system of vector functions $\{u_n\}_{n=1}^N$ is orthonormal in $\boldsymbol{L}_2(\Omega)$, then for each $j = 1, \dots, d$ the scalar family $\{u_n^j\}_{n=1}^N$ is suborthonormal.
We now take into account the divergence free condition and observe that

$$\xi \cdot \widehat{u}_n(\xi) = \xi \cdot (2\pi)^{-d/2} \int_\Omega e^{-i\xi x} u_n(x) dx$$
$$= i \int_\Omega u_n(x) \cdot \nabla_x e^{-i\xi x} dx = -i \int_\Omega e^{-i\xi x} \operatorname{div} u_n(x) dx = 0.$$

Let $\xi_0 = (a, 0, \dots, 0)$, $a \neq 0$. Then $\xi_0 \cdot \widehat{u}_n(\xi_0) = 0$. Hence, $\widehat{u}_n^1(\xi) = 0$ for $n = 1, \dots, N$ and

$$\sum_{n=1}^N |\widehat{u}_n(\xi_0)|^2 = \sum_{j=2}^d \sum_{n=1}^N |\widehat{u}_n^j(\xi_0)|^2 \leq (d-1)(2\pi)^{-d} |\Omega|.$$

The general case reduces to the above special case by an appropriate rotation.

Thus, $F_N(\xi) = \sum_{n=1}^N |\widehat{u}_n(\xi)|^2$ satisfies

$$\int_{\mathbb{R}^d} F_N(\xi)d\xi = N,$$

$$\int_{\mathbb{R}^d} |\xi|^2 F_N(\xi)d\xi = \sum_{n=1}^N \|\nabla u_n\|^2,$$

$$F_N(\xi) \le (d-1)(2\pi)^{-d}|\Omega|$$

We can now complete the proof of the theorem as in the case of the Dirichlet Laplacian:

$$\sum_{n=1}^N \|\nabla u_n\|^2 \ge \frac{d}{d+2}\left(\frac{(2\pi)^d}{(d-1)\cdot\omega_d|\Omega|}\right)^{2/d} \cdot N^{1+2/d}.$$

Remark

The constant is sharp. In fact, for the Stokes eigenvalue problem (at least in a smooth domain)

$$-\Delta u_k + \nabla p_k = \mu_k u_k,$$
$$\mathrm{div}\, u_k = 0, \;\; u_k|_{\partial\Omega} = 0,$$

$\|\nabla u_k\|^2 = \mu_k$ *and*

$$\|\nabla u_n\|^2 = \mu_n \sim \left(\frac{(2\pi)^d}{(d-1)\omega_d|\Omega|}\right)^{2/d} \cdot n^{2/d} \text{ as } n \to \infty.$$

Generalizations. Correction terms of subleading order

Theorem

Let $\Omega \subset \mathbb{R}^2$, $|\Omega| < \infty$. *Then*

$$\sum_{n=1}^{N} \lambda_n \geq \frac{2\pi}{|\Omega|} N^2 + \frac{1}{24} \frac{119}{120} \frac{|\Omega|}{\mathrm{I}} N \quad \textit{(Dirichlet Laplacian)},$$

$$\sum_{n=1}^{N} \mu_n \geq \frac{2\pi}{|\Omega|} N^2 + \frac{1}{48} \frac{239}{240} \frac{|\Omega|}{\mathrm{I}} N \quad \textit{(Stokes)}.$$

where $\mathrm{I} = \int_\Omega |x|^2 dx$ $\left(\mathrm{I} = \min_{a \in \mathbb{R}^2} \int_\Omega |x-a|^2 dx\right)$.

It is appropriate to mention here the following result (J.Kelliher). In two dimensions for a domain with smooth boundary

$$\mu_n > \lambda_n.$$

Generalizations. Dirichlet Laplacian in a domain on the sphere

Let us consider the 2D unit sphere $\mathbb{S}^2$ and the suface scalar Laplace–Beltrami operator

$$\Delta = \operatorname{div}\nabla.$$

Let $\Omega \subseteq \mathbb{S}^2$ be an arbitrary (curvilinear) domain on the sphere. We consider the Dirichlet eigenvalue problem

$$-\Delta\varphi_n = \lambda_n\varphi_n, \quad \varphi_n|_{\partial\Omega} = 0.$$

Theorem

$$\sum_{n=1}^{N}\lambda_n \ge \frac{2\pi}{|\Omega|}N\left(N - \frac{|\Omega|}{4\pi}\right).$$

Inequality turns into equality for $\Omega = \mathbb{S}^2$ and when N is a perfect square: $N = M^2$.

Stokes problem in a domain on the sphere

In the vector case we first define the Laplace operator acting on (tangent) vector fields on $\mathbb{S}^2$ as the Laplace–de Rham operator $-d\delta - \delta d$ identifying 1-forms and vectors. Then we have

$$\mathbf{\Delta} u = \nabla \operatorname{div} u - \operatorname{curl} \operatorname{curl} u,$$

where the operators $\nabla = \operatorname{grad}$ and div have the conventional meaning. The operator curl of a vector u is a scalar and for a scalar ψ, $\operatorname{curl} \psi$ is a vector:

$$\operatorname{curl} u := \operatorname{div} u^{\perp}, \qquad \operatorname{curl} \psi := \nabla^{\perp} \psi,$$

where in the local frame $u^{\perp} := (u_2, -u_1)$.

The Stokes problem reads

$$
\begin{aligned}
-\Delta u_k + \nabla p_k &= \mu_k u_k, \\
\operatorname{div} u_k = 0, \quad u_k|_{\partial\Omega} &= 0,
\end{aligned}
$$

Theorem

$$
\sum_{n=1}^{N} \mu_n \geq \frac{2\pi}{|\Omega|} N^2 + \frac{N}{2}.
$$

Inequality turns into equality for $\Omega = \mathbb{S}^2$ and when $N = M^2 - 1$.

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Thank you for your attention