Sobolev and spectral inequalities for orthonormal systems in the theory of attractors II
Leib–Thirring inequalities and attractors

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Lieb–Thirring inequaliteis (1976-2023)

We consider the Schrödinger operator

$$-\Delta - V,$$

on $L_2(\mathbb{R}^d)$. The potential $V = V(x) \geq 0$ decays sufficiently fast. We may assume that $V \in C_0^\infty(\mathbb{R}^d)$. The spectrum is the union of the continuous spectrum $[0, \infty)$ and (possibly) a discrete part consisting of negative eigenvalues $-\lambda_j < 0$ possibly accumulating only at zero. In fact, if $V \in C_0^\infty(\mathbb{R}^d)$, then there are finitely many negative eigenvalues. The Lieb–Thirring inequality bounds the sum of the γ-moments of λ_j in terms of the suitable L_p norm of the potential:

$$\sum \lambda_j^\gamma \leq L_{\gamma,d} \int_{\mathbb{R}^d} V(x)^{\gamma+d/2} dx.$$

The exponent $\gamma + d/2$ is uniquely defined by dimensional analysis, and

$$L_{\gamma,d} \geq L^{\mathrm{cl}}_{\gamma,d} := (2\pi)^{-d} \int_{\mathbb{R}^d} (1 - |\xi|^2)_+^\gamma d\xi = \frac{\Gamma(\gamma+1)}{(4\pi)^{d/2}\Gamma(\gamma + d/2 + 1)}.$$

The inequality holds for

$$d=1,\gamma\ge\frac{1}{2};\quad d=2,\gamma>0;\quad d\ge3,\gamma\ge0,$$

where the inequality in the case $d\ge3,\gamma=0$ is the celebrated CLR inequality.

What is known to date:

$$\begin{aligned} L_{\gamma,d} &= L^{\mathrm{cl}}_{\gamma,d}, \quad \gamma\ge\frac{3}{2};\\ L_{\gamma,d} &\le R_{1,1}L^{\mathrm{cl}}_{\gamma,d}, \quad 1\le\gamma<\frac{3}{2};\\ L_{\gamma,d} &\le 2R_{1,1}L^{\mathrm{cl}}_{\gamma,d}, \quad \frac{1}{2}\le\gamma<1, \end{aligned}$$

where

$$R_{1,1}\le 1.456\dots.$$

In what follows we shall concentrate on the important special case

$$\gamma=1$$

Theorem

The inequalty

$$\sum_{\lambda_j\le 0}|\lambda_j|\le L\int_{\mathbb{R}^d}V^{1+d/2}dx\quad \textit{for all}\quad 0\le V\in L_{1+d/2}(\mathbb{R}^d)$$

is equivalent to

$$\sum_{j=1}^N\|\nabla\psi_j\|^2\ge K\int_{\mathbb{R}^d}\left(\sum_{j=1}^N|\psi_j|^2\right)^{1+2/d}dx$$

for all N and all L_2-orthonormal $\{\psi_j\}_{j=1}^N\in H^1(\mathbb{R}^d)$

in the sense that optimal constants L and K are related by

$$\left(\left(1+\frac{d}{2}\right)L\right)^{1+2/d}\left(\left(1+\frac{2}{d}\right)K\right)^{1+d/2}=1.$$

1) $\Rightarrow$ Let $\{\psi_j\}_{j=1}^N \in H^1(\mathbb{R}^d)$ be orthonormal in L_2. Then by the variational principle

$$\sum_{j=1}^N \int_{\mathbb{R}^d} \left(|\nabla \psi_j|^2 - V|\psi_j|^2\right) dx \geq -\sum_{\lambda_n \leq 0} |\lambda_n| \geq -L \int_{\mathbb{R}^d} V^{1+d/2} dx,$$

or, setting $\rho(x) = \sum_{j=1}^N |\psi_j(x)|^2$

$$\sum_{j=1}^N \|\nabla \psi_j\|^2 \geq \int_{\mathbb{R}^d} (V\rho - LV^{1+d/2}) dx.$$

We maximize the right side by choosing $V = (\rho/(L(1+d/2)))^{2/d}$ and obtain

$$\sum_{j=1}^N \|\nabla \psi_j\|^2 \geq \frac{d/2}{(1+d/2)^{1+2/d}} L^{-2/d} \int_{\mathbb{R}^d} \rho^{1+2/d} dx.$$

2) $\Leftarrow$ Let ψ_n be normalized eigenfunctions of $-\Delta - V$ corresponding to negative eigenvalues $-\lambda_n$. Then

$$-\sum_n \lambda_n = \sum_n \int_{\mathbb{R}^d} (|\nabla\psi_n|^2 - V|\psi_n|^2)dx \geq \int_{\mathbb{R}^d} (K\rho^{1+2/d} - V\rho)dx$$
$$\geq KX - \left(\int_{\mathbb{R}^d} V^{1+d/2}dx\right)^{2/(2+d)} X^{d/(2+d)}.$$

where $X := \int_{\mathbb{R}^d} \rho^{1+2/d}dx$. We calculate the minimum of the RHS and obtain

$$-\sum_n \lambda_n \geq \min_{X\geq 0}\left(KX - \left(\int_{\mathbb{R}^d} V^{1+d/2}dx\right)^{2/(2+d)} X^{d/(2+d)}\right)$$
$$= -\frac{(d/2)^{d/2}}{(1+d/2)^{1+d/2}} K^{-d/2} \int_{\mathbb{R}^d} V^{1+d/2}dx.$$

A proof of dual LT inequality

Theorem

Let $\{\psi_j\}_{j=1}^N \in H^1(\mathbb{R}^d)$ *be orthonormal in* L_2. *Then*

$$\sum_{j=1}^N \|\nabla\psi_j\|^2 \ge K \int_{\mathbb{R}^d} (\rho(x))^{1+2/d}\, dx, \quad \rho(x) = \sum_{j=1}^N |\psi_j(x)|^2,$$

$$\textit{where} \quad K = \frac{d^2}{(d+4)(d+2)}(2\pi)^2\omega_d^{-2/d}.$$

Proof. For a positive self-adjoint operator A we define the spectral projections

$$P_E = \chi_{\mathrm{sp}(A)\cap[0,E]}, \qquad P_{E^\perp} = \chi_{\mathrm{sp}(A)\cap(E,\infty)}.$$

It follows from the spectral theorem that

$$A = \int_0^\infty P_{E^\perp}\, dE.$$

This especailly clear, if A has an orthonormal basis of eigenfunctions:

$$A = \sum_{j=1}^\infty \lambda_j (\cdot, y_j) y_j.$$

$$\sum_{j=1}^\infty \lambda_j a_j = \lambda_1(a_1+a_2+\dots)+(\lambda_2-\lambda_1)(a_2+a_3+\dots)+\dots = \int_0^\infty \sum_{\lambda_j \ge E} a_j dE,$$

it follows that

$$A = \int_0^\infty \sum_{\lambda_j \ge E} (\cdot, y_j) y_j dE = \int_0^\infty P_E^\perp dE.$$

For $-\Delta$ on $\mathbb{R}^d$ we therefore have

$$-\Delta = \int_0^\infty P_{E^\perp}\, dE, \quad P_E = \chi_{[0,E]}, \qquad P_{E^\perp} = \chi_{(E,\infty)} = Id - P_E.$$

Given the orthonormal family $\{\psi_j\}_{j=1}^N$, let Γ be the finite rank orthogonal projection

$$\Gamma = \sum_{j=1}^N (\cdot, \psi_j)\psi_j.$$

Thanks to the representation of $-\Delta$

$$\|\nabla\psi\|^2 = (-\Delta\psi, \psi) = \int_0^\infty (P_E^\perp\psi, \psi)dE = \int_0^\infty \|P_E^\perp\psi\|^2 dE =$$

$$= \int_0^\infty \int_{\mathbb{R}^d} |P_E^\perp\psi(x)|^2 dxdE = \int_{\mathbb{R}^d}\int_0^\infty |P_E^\perp\psi(x)|^2 dEdx.$$

Therefore

$$\sum_{j=1}^N \|\nabla\psi_j\|^2 = \int_{\mathbb{R}^d}\int_0^\infty \rho_E(x)dEdx,$$

where

$$\rho_E(x) := \sum_{j=1}^N |P_E^\perp\psi_j(x)|^2.$$

Now let B be a small ball around a point $a \in \mathbb{R}^d$ with area $|B|$ and let χ_B be the corresponding characteristic function. Then denoting by $\|\cdot\|_{\mathrm{HS}}$ the Hilbert–Schmidt norm and using the triangle inequality for it we obtain

$$\left(\int_B \rho(x)dx\right)^{1/2} = \|\Gamma\chi_B\|_{\mathrm{HS}} \le \|\Gamma P_E\chi_B\|_{\mathrm{HS}} + \|\Gamma P_E^{\perp}\chi_B\|_{\mathrm{HS}} = \\ = \|\Gamma P_E\chi_B\|_{\mathrm{HS}} + \left(\int_B \rho_E(x)dx\right)^{1/2}.$$

Using that Γ is bounded ($\|\Gamma\| = 1$) and then the fact that both χ_B and P_E are projections, we find that

$$\|\Gamma P_E \chi_B\|_{\mathrm{HS}}^2 \le \|P_E \chi_B\|_{\mathrm{HS}}^2 = \mathrm{Tr}(P_E \chi_B) =$$
$$= |B| \int_{|\xi|^2 < E} \frac{d\xi}{(2\pi)^d} = |B|(2\pi)^{-d} \omega_d E^{d/2}.$$

where we used the fact that P_E has the integral kernel

$$(2\pi)^{-d} \int_{|\xi|^2 < E} e^{i\xi \cdot (x-y)} d\xi.$$

This gives

$$\left(\int_B \rho(x)dx\right)^{1/2} \le \left(|B|(2\pi)^{-d}\omega_d E^{d/2}\right)^{1/2} + \left(\int_B \rho_E(x)dx\right)^{1/2}.$$

Dividing by $|B|^{1/2}$ and letting $|B| \to 0$ we obtain

$$\rho(a)^{1/2} \le (2\pi)^{-d/2}\omega_d^{1/2} E^{d/4} + \rho_E(a)^{1/2},$$

Since $\rho_E \geq 0$ we have actually shown that for almost every $a \in \mathbb{R}^d$

$$\rho_E(a) \geq \left(\rho(a)^{1/2} - (2\pi)^{-d/2}\omega_d^{1/2}E^{d/4}\right)_+^2.$$

Integrating with respect to E we obtain

$$\int_0^\infty \rho_E(a)dE \geq \int_0^\infty \left(\rho(a)^{1/2} - (2\pi)^{-d/2}\omega_d^{1/2}E^{d/4}\right)_+^2 dE =$$

$$\rho(a)^{1+2/d}\int_0^\infty \left(1 - (2\pi)^{-d/2}\omega_d^{1/2}\sigma^{d/4}\right)_+^2 d\sigma =$$

$$\rho(a)^{1+2/d}\frac{d^2}{(d+4)(d+2)}(2\pi)^2\omega_d^{-2/d} =: \rho(a)^{1+2/d}\cdot K.$$

Finally, integration over $\mathbb{R}^d$ completes the proof:

$$\sum_{j=1}^N \|\nabla\psi_j\|^2 = \int_{\mathbb{R}^d}\int_0^\infty \rho_E(x)dEdx \geq K\int_{\mathbb{R}^d}\rho(x)^{1+d/2}dx.$$

Remark

The constant K in the Theorem is not the best available to date. For example, for $d = 2$ we obtained

$$\int_{\mathbb{R}^2} \rho(x)^2 dx \le k \sum_{j=1}^{N} \|\nabla \psi_j\|^2, \qquad \rho(x) = \sum_{j=1}^{N} |\psi_j(x)|^2,$$

$$k \le \frac{3}{2\pi} \qquad \left(k = \frac{1}{K}\right)$$

while the best estimate to date is

$$k \le \frac{1}{2\pi} \cdot R_{1,1} = \frac{1}{2\pi} \cdot 1.456 \ldots$$

$d = 2$, $\operatorname{div} u = 0$

Theorem

Suppose that vector functions $u_1, \dots, u_m \in \boldsymbol{H}_0^1(\Omega)$ make up an orthonormal family in $\boldsymbol{L}_2(\Omega)$, $\Omega \subseteq \mathbb{R}^2$. Suppose that $\operatorname{div} u_j = 0$ (or $\operatorname{curl} u_j = 0$) for $j = 1, \dots, m$. Then the following inequalities hold:

$$\int_\Omega \rho(x)^2 dx \ \le \ k_2^{sol} \sum_{j=1}^m \|\operatorname{curl} u_j\|^2, \quad \text{if } \operatorname{div} u_j = 0,$$

$$\int_\Omega \rho(x)^2 dx \ \le \ k_2^{pot} \sum_{j=1}^m \|\operatorname{div} u_j\|^2, \quad \text{if } \operatorname{curl} u_j = 0,$$

where $\rho(x) = \sum_{j=1}^m |u_j(x)|^2$ and the best constants k_2^{sol} and k_2^{pot} satisfy the relation

$$k_2^{sol} = k_2^{pot} \le k = 4L_{1,2}.$$

Ladyzhenskaya inequality

For $\psi \in H_0^1(\Omega)$, $\Omega \subseteq \mathbb{R}^2$ it holds

$$\|\psi\|^4_{L_4(\Omega)} \le c_{\mathrm{Lad}} \|\psi\|^2 \|\nabla\psi\|^2,$$

while the LT inequality with $N = 1$ goes over to

$$\|\psi\|^4_{L_4(\Omega)} \le k \|\psi\|^2 \|\nabla\psi\|^2.$$

The value of c_{Lad} is known

$$c_{\mathrm{Lad}} = \frac{1}{\pi \cdot 1.8622\dots} = \frac{1}{2\pi} \cdot 1.073\dots$$

Since clearly $k \ge c_{\mathrm{Lad}}$, it follows that

$$\frac{1}{2\pi} \cdot 1.073\dots \le k \le \frac{1}{2\pi} \cdot R_{1,1} = \frac{1}{2\pi} \cdot 1.456\dots$$

LT on $\mathbb{S}^2$, $\mathbb{T}^d$, $\mathbb{H}^d$

Theorem

Let $\mathbb{M} = \mathbb{S}^2$ *or* $\mathbb{M} = \mathbb{T}^2 = [0, L] \times [0, L]$. *Let* $\{\psi_n\}_{n=1}^N \in \dot{H}^1(\mathbb{M})$ *be an orthonormal system of zero-mean functions in* $L_2(\mathbb{M})$, $(\psi_n, 1) = 0$. *Then* $\rho = \sum_{n=1}^N |\psi_n|^2$ *satisfies*

$$\int_{\mathbb{M}} \rho^2 dm \le k \sum_{n=1}^N \|\nabla \psi_n\|^2, \qquad k \le \frac{3\pi}{32} = 0.477\dots.$$

If $\Omega \subseteq \mathbb{S}^2$ *and* $\{u_n\}_{n=1}^N \in \boldsymbol{H}_0^1(\Omega)$ *is an orthonormal system of (tangent) vector functions in* $\boldsymbol{L}_2(\Omega)$ *and* $\operatorname{div} u_n = 0$, *then* $\rho(x) = \sum_{n=1}^N |u_n(x)|^2$ *satisfies*

$$\int_{\Omega} \rho^2 dS \le k \sum_{n=1}^N \|\operatorname{curl} \psi_n\|^2, \qquad k \le \frac{3\pi}{32} = 0.477\dots.$$

Remark

If $\mathbb{T}^2 = \mathbb{T}^2_\alpha = [0, L/\alpha] \times [0, L]$, $\alpha \le 1$, *then*

$$k \to \frac{k}{\alpha}.$$

If we impose the condition that ψ_n*'s satisfy*

$$\int_0^{2\pi} \psi_n(x_1, t)dt = 0 \quad \text{for all} \quad x_1 \in [0, 2\pi/\alpha],$$

then uniformly with respect to α

$$\int_{\mathbb{T}^2_\alpha} \rho^2 dx \le \frac{\pi}{6} \sum_{n=1}^{N} \|\nabla \psi_n\|^2.$$

Manifold with constant negative curvature

Let $\mathbb{H}^d$, $d \geq 2$, be the upper half-space

$$\mathbb{H}^d = \{(x, y) : \ x \in \mathbb{R}^{d-1}, \ y \in \mathbb{R}_+\}$$

with the Poincare metric $ds^2 = y^{-2}(dx^2 + dy^2)$. We consider the self-adjoint Laplace–Beltrami operator in $L^2\left(\mathbb{H}^d, \frac{dx\,dy}{y^d}\right)$

$$-\Delta_h = -y^d \frac{\partial}{\partial y} y^{2-d} \frac{\partial}{\partial y} - y^2 \sum_{n=1}^{d-1} \frac{\partial^2}{\partial x_n^2}.$$

The continuous spectrum of $-\Delta_h$ covers the half-line

$$\left[\frac{(d-1)^2}{4}, \ \infty\right].$$

Therefore the discrete spectrum $\{\lambda_k\}$ of the Schrödinger operator

$$-\Delta_h - V,$$

acting in $L^2\left(\mathbb{H}^d, \frac{dx\,dy}{y^d}\right)$ lies below $(d-1)^2/4$. We write $\{\lambda_k\}$ as follows

$$\lambda_k = \frac{(d-1)^2}{4} - \mu_k, \quad -\mu_k \le 0.$$

Theorem

Let $V \ge 0$ and $\gamma \ge 1/2$. Then

$$\sum \mu_k^\gamma \le L_{\gamma,d} \int_{\mathbb{H}^d} V(x,y)^{\gamma+d/2} \frac{dx\,dy}{y^d},$$

where

$$L_{\gamma,d} = \begin{cases} L^{\mathrm{cl}}_{\gamma,d}, & \gamma \ge 3/2, \\ R_{1,1} L^{\mathrm{cl}}_{\gamma,d} & 1 \le \gamma < 3/2, \\ 2R_{1,1} L^{\mathrm{cl}}_{\gamma,d} & 1/2 \le \gamma < 1, \end{cases}$$ *where, as before, $R_{1,1} = 1.456\dots$.*

Applications: 2D NS

Berezin–Li&Yau inequalities for the Stokes operator and the LT inequalities for divergence free orthonormal vector functions make it possible to write down the estimates for the fractal dimension of the 2D Navier–Stokes system in the explicit form.

$$\partial_t u + \nabla_u u + \nabla p - \nu \Delta u = f, \quad \operatorname{div} u = 0, \quad u|_{\partial\Omega} = 0.$$

Here $\Omega \subset \mathbb{R}^2$ or $\Omega \subset \mathbb{S}^2$.

Theorem

$$\dim_F \mathscr{A} \le \frac{1}{2\pi}\left(\frac{k_{\mathrm{LT}}}{2}\right)^{1/2} \cdot \frac{\|f\||\Omega|}{\nu^2} = \begin{cases} 0.055 \cdot \frac{\|f\||\Omega|}{\nu^2}, & \Omega \subset \mathbb{R}^2, \\ 0.062 \cdot \frac{\|f\||\Omega|}{\nu^2}, & \Omega \subset \mathbb{S}^2. \end{cases}$$

We recall that

$$k_{\mathrm{LT}}(\mathbb{R}^2) \le \frac{1}{2\pi} \cdot 1.456, \quad k_{\mathrm{LT}}(\mathbb{S}^2) \le \frac{3\pi}{32}.$$

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Thank you for your attention