Sobolev and spectral inequalities for orthonormal systems in the theory of attractors III Inequalities for systems with orthonormal derivatives and applications

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Damped/driven regularized Euler equations

We shall be dealing with the following approximation of the damped Euler system, the so-called inviscid damped Euler–Bardina model

$$\begin{cases} \partial_t u + (\bar u, \nabla)\bar u + \gamma u + \nabla p = g, \\ \operatorname{div} u = 0, \quad u(0) = u_0, \quad \bar u = (1 - \alpha\Delta)^{-1} u. \end{cases}$$

The system is studied for $d = 2, 3$
1) on the torus $\Omega = \mathbb{T}^d = [0, L]^d$ with standard zero mean condition;
2) in $\Omega \subseteq \mathbb{R}^d$;
3) on the sphere or in a domain on it $\Omega \subseteq \mathbb{S}^2$;
3) if $\Omega \subsetneq \mathbb{R}^d$ or $\Omega \subsetneq \mathbb{S}^2$, then $\bar u|_{\partial\Omega} = 0$ and $\bar u$ is recovered from u by solving the Stokes problem

$$\begin{cases} (1 - \alpha\Delta)\bar u + \nabla q = u, \\ \operatorname{div} \bar u = 0, \quad \bar u|_{\partial\Omega} = 0. \end{cases}$$

Here $\alpha = \alpha' L^2$ and $\alpha' > 0$ is a small dimensionless parameter, so that $\bar u$ is a smoothed (filtered) vector field.

The phase space with respect to $\bar{u}$ is the Sobolev space $\mathbf{H}^1$ with divergence free condition

$$\bar{u} \in \mathbf{H}^1 := \begin{cases} \dot{\mathbf{H}}^1(\mathbb{T}^d), & x \in \mathbb{T}^d,\ \int_{\mathbb{T}^d} \bar{u}(x)dx = 0, \\ \mathbf{H}^1(\mathbb{R}^d), & x \in \mathbb{R}^d, \\ \mathbf{H}^1_0(\Omega), & x \in \Omega \subsetneq \mathbb{R}^d, \mathbb{S}^2 \end{cases} \qquad \operatorname{div} \bar{u} = 0,$$

and in terms of u, respectively,
$u \in \mathbf{H}^{-1} := (1 - \Delta)\mathbf{H}^1 = \mathbf{H}^{-1} \cap \{\operatorname{div} u = 0\}$.
We write the equation as an evolution equation in $\mathbf{H}^1$:

$$\partial_t \bar{u} + B(\bar{u}, \bar{u}) + \gamma \bar{u} = \bar{g},$$
$$\operatorname{div} \bar{u} = 0, \quad \bar{u}(0) = \bar{u}_0, \quad u = (1 - \alpha\Delta)\bar{u},$$

where

$$B(\bar{u}, \bar{v}) = (1 - \alpha\Pi\Delta)^{-1}\Pi((\bar{u}, \nabla)\bar{v}),$$

Π — is the Helmholtz–Leray projection, $\Pi\Delta$ — is the Stokes operator.

Sharp upper bounds for the dimension of attractor

Theorem

Let $d = 2$. In each case of BC the system possesses a global attractor $\mathscr{A} \Subset \mathbf{H}^1$ with finite fractal dimension satisfying

$$\dim_F \mathscr{A} \le \frac{1}{8\pi} \cdot \begin{cases} \dfrac{1}{\alpha\gamma^4} \min\left(\| \operatorname{rot} g\|^2_{L^2}, \ \dfrac{\|g\|^2_{L^2}}{2\alpha} \right), & x \in \mathbb{T}^2, \mathbb{R}^2, \mathbb{S}^2 \\ \dfrac{\|g\|^2_{L^2}}{2\alpha^2\gamma^4}, & x \in \Omega \subsetneq \mathbb{R}^2, \mathbb{S}^2. \end{cases}$$

In the 3D case the estimates in all three cases look formally the same

$$\dim_F \mathscr{A} \le \frac{1}{12\pi} \frac{\|g\|^2_{L^2}}{\alpha^{5/2}\gamma^4}, \quad x \in \mathbb{T}^3, \ x \in \mathbb{R}^3, \ x \in \Omega \subsetneq \mathbb{R}^3 .$$

Kolmogorov flows and lower bounds

The lower bounds are based on the on the instability analysis of the generalized Kolmogorov flows. Let

$$g_s(x_2) = (\gamma\lambda(s)\sin sx_2, 0)^T, \qquad g_s(x_3) = (\gamma\lambda(s)\sin sx_3, 0, 0)^T$$

be the right-hand sides in our system on $\mathbb{T}^2 = [0, 2\pi]^2$ and $\mathbb{T}^3 = [0, 2\pi]^3$, respectively. Here $s \in \mathbb{N}$, $s \gg 1$, and λ is the amplitude. The corresponding staionary solutions are

$$u_s(x_2) = (\lambda(s)\sin sx_2, 0)^T, \qquad u_s(x_3) = (\lambda(s)\sin sx_3, 0, 0)^T,$$

since the nonlinear term vanishes on them.

Theorem

For $\lambda \geq \lambda(s)$, where

$$\lambda(s) = c_1 \gamma \frac{(1+\alpha s^2)^2}{s},$$

and c_i are absolute (effectively computable) constants the stationary solutions are unstable and

$$\dim \mathcal{M}^{\mathrm{un}}(u_s) \geq c_2 s^2,\ d = 2; \qquad \dim \mathcal{M}^{\mathrm{un}}(u_s) \geq c_3 s^3,\ d = 3.$$

Corollary

$$\dim_F \mathscr{A} \geq c_6 \begin{cases} \max\left(\frac{\|\operatorname{rot} g_s\|^2_{L^2}}{\alpha\gamma^4}, \frac{\|g_s\|^2_{L^2}}{\alpha^2\gamma^4}\right), & x \in \mathbb{T}^2, \\ \frac{\|g_s\|^2_{L^2}}{\alpha^{5/2}\gamma^4}, & x \in \mathbb{T}^3. \end{cases}$$

Systems with orthonormal derivatives. E. Lieb 1983

Estimating the N-trace of the linearized operator we naturally come to systems of functions $\{\psi_j\}_{j=1}^N \in H^1$ that are orthonormal wrt

$$m^2(\psi_i, \psi_j) + (\nabla\psi_i, \nabla\psi_j) = \delta_{ij}, \qquad m > 0 \qquad (*)$$

and we need bounds for the L_p-norms of the function

$$\rho(x) = \sum_{j=1}^{N} |\psi_j(x)|^2.$$

Theorem

Let $\{\psi_j\}_{j=1}^N \in H^1(\mathbb{R}^d)$ *satisfy* $(*)$*. We set* $\rho(x) = \sum_{j=1}^N |\psi_j(x)|^2$*. Then*

1) $d = 1 \quad \|\rho\|_{L_\infty} \le L/m,$

2) $d = 2 \quad \|\rho\|_{L_p} \le B_p m^{-2/p} N^{1/p}, \quad B_p \le \left(\frac{1}{4\pi}\right)^{(p-1)/p} (p-1)^{(p-1)/p}$

3) $d \ge 3 \quad \|\rho\|_{L_{d/(d-2)}} \le A_d N^{(d-2)/d}, \quad m \ge 0.$

where

$$L = 1, \quad B_p \le \left(\frac{1}{4\pi}\right)^{(p-1)/p} (p-1)^{(p-1)/p}, \quad A_d \le L_{0,d}^{2/d} \frac{d}{d-2}$$

and $L_{0,d}$ *is the constant in the Cwikel–Lieb–Rozenblum inequality*

$$N(0, -\Delta - V) \le L_{0,d} \int_{\mathbb{R}^d} V(x)^{d/2} dx.$$

For $d = 1$ we have the inequality $\|\psi\|_{L_\infty}^2 \le \|\psi\|\|\psi'\|$, which saturates at $\psi(x) = e^{|x|}$. Therefore for $u(x) = \sum_{j=1}^N \xi_j\psi_j(x)$ it holds for a fixed $x \in \mathbb{R}$

$$\left(\sum_{j=1}^N \xi_j\psi_j(x)\right)^2 \le \|u\|_{L_\infty}^2 \le \|u\|\|u'\|$$

$$= \frac{1}{m}\left(\sum_{i,j=1}^N (\xi_i\xi_j(m\psi_i, m\psi_j)\right)^{1/2}\left(\sum_{i,j=1}^N \xi_i\xi_j(\psi_i', \psi_j')\right)^{1/2} \le$$

$$\frac{1}{m}\left(\sum_{j=1}^N \xi_j^2\right)^{1/2}\left(\sum_{j=1}^N \xi_j^2\right)^{1/2} = \frac{1}{m}\sum_{j=1}^N \xi_j^2,$$

since both systems $\{m\psi_j\}_{j=1}^N, \{\psi_j'\}_{j=1}^N$ are suborthonormal. Setting $\xi_i = \psi_i(x)$ gives

$$\sum_{j=1}^N \psi_j(x)^2 \le \frac{1}{m}.$$

For $d = 2$ we have

$$\left\| \sum_{j=1}^{N} |\psi_j|^2 \right\|_{L_p} \le \left(\frac{1}{4\pi}\right)^{(p-1)/p} (p-1)^{(p-1)/p} m^{-2/p} N^{1/p}.$$

Setting $N = 1$ here, this inequality gives for a function $\psi \in H^1(\mathbb{R}^2)$ and

$$\widetilde{\psi} = \frac{\psi}{(m^2\|\psi\|^2 + \|\nabla\psi\|^2)^{1/2}}$$

the following multiplicative inequality

$$\|\psi\|_{L^q} \le \left(\frac{1}{4\pi}\right)^{(q-2)/2q} \left(\frac{q}{2}\right)^{1/2} \|\psi\|^{2/q} \|\nabla\psi\|^{1-2/q}, \quad q = 2p \in [2, \infty),$$

which gives nothing new in $\mathbb{R}^2$ and is known even with a better constant.

The Sobolev inequality in $\mathbb{R}^2$ ($\mathbb{R}^d$)

$$\|f\|_{L_q}^2 \le C_q(\|f\|^2 + \|\nabla f\|^2), \quad q \ge 2, \quad \|\cdot\| := \|\cdot\|_{L_2}.$$

can be written in the Garliardo–Nirenberg form by scaling $x \to \varepsilon x$:

$$\|f\|_{L_q}^2 \le C_q(\varepsilon^{\frac{4}{q}-2}\|f\|^2 + \varepsilon^{\frac{4}{q}}\|\nabla f\|^2).$$

Optimizing in ε we obtain

$$\|f\|_{L_q}^2 \le C_q \frac{q/2}{(q/2-1)^{1-2/q}} \|f\|^{\frac{4}{q}} \|\nabla f\|^{2-\frac{4}{q}}.$$

To go other way round we use the arithmetic-geometric mean inequality

$$a^\theta b^{1-\theta} \le \theta a + (1-\theta) b,$$

So that Sobolev inequalities in additive and multiplicative form are equivalent.

Theorem (Sh.M. Nasibov, E.Lieb–M.Loss)

For $f \in H^1(\mathbb{R}^2)$ *it holds*

$$\|f\|_{L^q} \le \mathrm{c}_q \|f\|^{2/q} \|\nabla f\|^{1-2/q}, \quad q \in [2, \infty).$$

where

$$\mathrm{c}_q \le \left(\frac{1}{4\pi}\right)^{(q-2)/2q} \left(\frac{q}{2}\right)^{1/2},$$

or even

$$\mathrm{c}_q \le \left(\frac{1}{4\pi}\right)^{(q-2)/2q} \left(\frac{q}{2}\right)^{1/2} \frac{q^{(q-2)/q}}{(q-1)^{(q-1)/q}},$$

and

$$\frac{q^{(q-2)/q}}{(q-1)^{(q-1)/q}} < 1 \quad q \in (2, \infty) \quad \textit{and} \quad \lim_{q \to 2, q \to \infty} \frac{q^{(q-2)/q}}{(q-1)^{(q-1)/q}} = 1.$$

Proof. By Huasdorff–Young inequality

$$\|f\|_{L^q} \le (2\pi)^{-\frac{q-2}{q}} \|\widehat{f}\|_{L^{q'}} =$$

$$= (2\pi)^{-\frac{q-2}{q}} \|(|\xi|^2 + m^2)^{1/2}\widehat{f}(\xi)(|\xi|^2 + m^2)^{-1/2}\|_{L^{q'}} \le$$

$$\le (2\pi)^{-\frac{q-2}{q}} \|(|\xi|^2 + m^2)^{1/2}\widehat{f}\|_{L^2} \||\xi|^2 + m^2)^{-1/2}\|_{L^{\frac{2q}{q-2}}}.$$

Calculating the integral (note that $q/(q-2) > 1$)

$$\|(|\xi|^2 + m^2)^{-1/2}\|_{L^{\frac{2q}{q-2}}}^{\frac{2q}{q-2}} = \int_{\mathbb{R}^2} \frac{d\xi}{(|\xi|^2 + m^2)^{\frac{q}{q-2}}} = \pi \frac{q-2}{2} m^{-\frac{4}{q-2}},$$

we obtain

$$\|f\|_{L^q}^2 \le (8\pi)^{-\frac{q-2}{q}} (q-2)^{\frac{q-2}{q}} m^{-\frac{4}{q}} \left(\|\nabla f\|^2 + m^2\|f\|^2\right).$$

Choosing optimal m

$$m^2 = \frac{2}{q-2} \frac{\|\nabla f\|^2}{\|f\|^2}$$

gives the first estimate.

The second estimate follows from the sharp form of the Hausdorff-Young inequality (Babenko–Beckner)

$$\|f\|_{L^p(\mathbb{R}^d)} \le \left((2\pi)^{\frac{1}{p}-\frac{1}{q}}\ \frac{q^{\frac{1}{q}}}{p^{\frac{1}{p}}}\right)^{d/2} \|\widehat{f}\|_{L^q(\mathbb{R}^d)}, \quad p \ge 2, \quad \frac{1}{p}+\frac{1}{q}=1$$

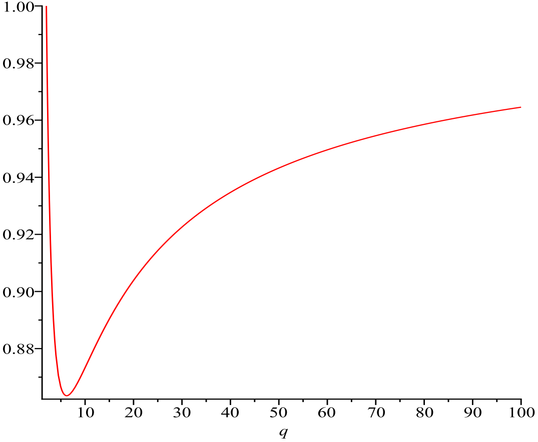

Figure: Graph of the Babenko–Beckner correction term $q \to \frac{q^{(q-2)/q}}{(q-1)^{(q-1)/q}}$.

Sharp constant M.Weinstein 1983

The sharp constant c_q is given by

$$\mathrm{c}_q = \left(\frac{q}{2}\right)^{\frac{1}{q}} \cdot \frac{1}{\|\varphi\|_{L^2(\mathbb{R}^2)}^{1-\frac{2}{q}}},$$

where φ is the smooth positive decreasing solution of the Euler–Lagrange equation

$$\frac{q-2}{2}\Delta\varphi - \varphi + \varphi^{q-1} = 0, \quad x \in \mathbb{R}^2$$

with minimal L^2-norm. Then $\varphi(x) = \psi(|x|) = \psi(r)$, and

$$\psi'' + \frac{1}{r}\psi' = \frac{2}{q-2}\left(\psi - \psi^{q-1}\right), \quad r \in (0,\infty), \quad \psi(0) = a,\ \psi'(0) = 0.$$

Here $\psi'(0) = 0$ in view of smoothness in x, and the problem is to find a (uniquie) initial value a, for which there exists the solution with finite L_2-norm.

$q = 4$ Ladyzhenskaya inequality

$$\|f\|_{L_4} \le c_4 \|f\|^{1/2} \|\nabla f\|^{1/2}.$$

The sharp constant

$$\mathrm{c}_4 = \left(\frac{1}{\pi \cdot 1.86225\ldots}\right)^{\frac{1}{4}} = \left(\frac{1}{5.8504309}\right)^{\frac{1}{4}},$$

while both estimates in the theorem give

$$\mathrm{c}_4 < \left(\frac{1}{\pi}\right)^{\frac{1}{4}}, \quad \mathrm{c}_4 < \left(\frac{16}{\pi \cdot 27}\right)^{\frac{1}{4}} = \left(\frac{1}{5.301437604}\right)^{\frac{1}{4}}.$$

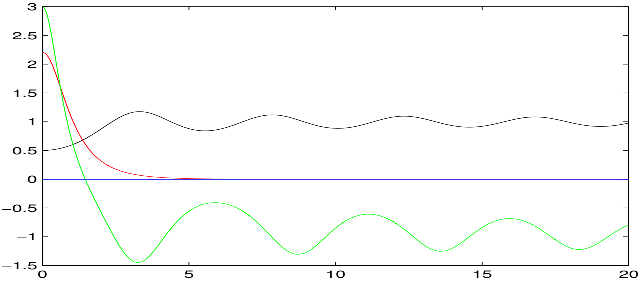

Figure: Solutions: $\psi(0) = a^* = 2.206200864\dots$ (red), $\psi(0) = 3$ (green) ? $\psi(0) = 0.5$ (black), $q = 4$.

$$\psi'' + \frac{1}{r}\psi' = \left(\psi - \psi^3\right), \quad r \in (0, \infty), \quad \psi(0) = a, \ \psi'(0) = 0.$$

$$\|\psi^*\|_r^2 = \int_0^\infty \psi^*(r)^2 r\, dr = 1.8622556\dots .$$

$\mathbb{T}^2 = [0, L]_{\mathrm{per}} \times [0, L]_{\mathrm{per}}$

Theorem

It holds for $\psi \in \dot{H}^1(\mathbb{T}^2)$

$$\|\psi\|_{L^q} \le \mathrm{c}_q \|\psi\|^{2/q} \|\nabla\psi\|^{1-2/q}, \quad q \in [2, \infty), \quad \int_{\mathbb{T}^2} \psi(x)dx = 0$$

where (by notational definition) sharp constant c_q *satisfies*

$$\mathrm{c}_q \le \left(\frac{1}{4\pi}\right)^{(q-2)/2q} \left(\frac{q}{2}\right)^{1/2}.$$

Proof. $L = 2\pi$. The Fourier series $\varphi(x) = \sum_{n\in\mathbb{Z}_0^2} a_n e^{ix\cdot n}$. Parseval's identity and thye fact that $|e^{ix\cdot n}| = 1$ give

$$\|\varphi\|_{L^2} = 2\pi\|a\|_{l^2}, \qquad \|\varphi\|_{L^\infty} \le \|a\|_{l^1}.$$

By the Riesz–Thorin theorem this gives the Hausdorff–Young inequality for the Fourier series

$$\|\varphi\|_{L^p} \le (2\pi)^{2/p}\|a\|_{l^q}, \quad \frac{1}{p} + \frac{1}{q} = 1, \quad p \ge 2.$$

For an $m > 0$

$$\begin{aligned}\|\varphi\|_{L^p} \le (2\pi)^{2/p}\|a\|_{l^q} = \\ = (2\pi)^{2/p}\left\|(m^2 + |n|^2)^{-1/2} \cdot (m^2 + |n|^2)^{1/2} a_n\right\|_{l^q} \le \\ \le (2\pi)^{2/p}\left(\sum_{n\in\mathbb{Z}_0^2} \frac{1}{(m^2 + |n|^2)^{r/2}}\right)^{1/r} \|(m^2 + |n|^2)^{1/2}|a_n|\|_{l^2},\end{aligned}$$

where $\frac{1}{r} + \frac{1}{2} = \frac{1}{q}$, so that $\frac{r}{2} = 1 + \frac{2}{p-2} > 1$.

Using the key estimate for the series over $\mathbb{Z}_0^2$

$$\left(\sum_{n\in\mathbb{Z}_0^2}\frac{1}{(m^2+|n|^2)^{\frac{r}{2}}}\right)^{\frac{1}{r}} < \left(\frac{\pi}{(r/2-1)m^{2(\frac{r}{2}-1)}}\right)^{\frac{1}{r}} = \left(\frac{\pi(p-2)}{2}\right)^{\frac{p-2}{2p}} m^{-\frac{2}{p}}$$

and taking into account

$$\|(m^2+|n|^2)^{1/2}|a_n|\|_{l^2}^2 = \frac{1}{4\pi^2}\|(m^2-\Delta)^{1/2}\varphi\|^2 = \frac{1}{4\pi^2}(m^2\|\varphi\|_{L^2}^2+\|\nabla\varphi\|_{L^2}^2)$$

we obtain

$$\|\varphi\|_{L^p} \le \left(\frac{p-2}{8\pi}\right)^{\frac{p-2}{2p}} m^{-2/p}\left(m^2\|\varphi\|_{L^2}^2+\|\nabla\varphi\|_{L^2}^2\right)^{1/2}, \quad p\ge 2.$$

Changing p to $2p$, we obtain

$$\|\varphi\|_{L^{2p}}^2 \le \left(\frac{p-1}{4\pi}\right)^{\frac{p-1}{p}}\left(m^{2-2/p}\|\varphi\|^2+m^{-2/p}\|\nabla\varphi\|^2\right), \quad p\ge 1,$$

and the result follows by taking the optimal m.

Lemma

For $p > 1$ and $m \geq 0$

$$I_p(m) := \frac{(p-1)m^{2(p-1)}}{\pi} \sum_{n \in \mathbb{Z}_0^2} \frac{1}{(m^2 + |n|^2)^p} < 1.$$

It suffices to show that

$$I_p(\infty) = 1 \quad \text{and} \quad \frac{d}{dm} I_p(m) > 0 \quad \text{for } m > 0.$$

Ovserve that

$$\frac{(p-1)m^{2(p-1)}}{\pi} \int_{\mathbb{R}^2} \frac{dx}{(m^2 + |x|^2)^p} = 1.$$

$$I_p(m) = \frac{(p-1)m^{2(p-1)}}{\pi\Gamma(p)} \int_0^\infty x^{p-1} e^{-m^2 x} \left(\theta_3^2(e^{-x}) - 1\right) dx,$$

where $\theta_3(q)$ is the Jacobi theta function

$$\theta_3(q) = \sum_{n=-\infty}^{\infty} q^{n^2}.$$

The key functional identity

$$\varphi(x) = \frac{\varphi(x^{-1})}{\sqrt{x}}, \qquad \varphi(x) := \theta_3(e^{-\pi x}) = \sum_{n=-\infty}^{\infty} e^{-\pi x n^2}.$$

Write the series in terms of φ and change $m^2 x \to x$:

$$I_p(m) = \frac{(p-1)}{\pi\Gamma(p)} \int_0^\infty x^{p-1} e^{-x} \left(m^{-2}\varphi^2\left(\frac{x}{\pi m^2}\right) - m^{-2}\right) dx =$$

$$= \frac{(p-1)}{\pi\Gamma(p)} \int_0^\infty x^{p-1} e^{-x} \left(\frac{\pi}{x}\varphi^2\left(\frac{\pi m^2}{x}\right) - m^{-2}\right) dx.$$

This is suitable for differentiation

$$\frac{d}{dm} I_p(m) = \frac{2(p-1)}{\pi\Gamma(p)m^3} \int_0^\infty x^{p-1} e^{-x} \left(2y^2\varphi(y)\varphi'(y) + 1\right) dx,$$

where $y = y(x) := \pi m^2/x$. We shall prove monotonicity if we show that

$$2y^2\varphi(y)\varphi'(y) + 1 \geq 0, \quad y \in \mathbb{R}_+.$$

Recall that

$$\varphi(x) := \theta_3(e^{-\pi x}) = \sum_{n=-\infty}^{\infty} e^{-\pi x n^2}.$$

This condition is independent of p !

Since $\varphi'(y) \to 0$ as $y \to \infty$ very fast it is easy to verify this condition for large y ($y > \pi$). For small $y \leq \pi$ we use $\varphi(y) = \frac{\varphi(y^{-1})}{\sqrt{y}}$.

Elongated torus $\mathbb{T}^2_\alpha = [0, L] \times [0, L/\alpha]$, $\alpha \ll 1$

Let $c_q = c_q(1)$ be the constant in

$$\|f\|_{L^q} \le \mathrm{c}_q \|f\|^{2/q} \|\nabla f\|^{1-2/q}, \quad q \in [2, \infty).$$

And let $1/\alpha$ be interger. Then on $\mathbb{T}^2_\alpha$

$$c_q(\alpha) = c_q(1) \left(\frac{1}{\alpha}\right)^{\frac{q-2}{2q}}.$$

$\mathbb{S}^2$

Interpolation inequality on $\mathbb{S}^d$ (W.Beckner 1993):

$$\frac{q-2}{d}\int_{\mathbb{S}^d}|\nabla\varphi|^2 d\mu+\int_{\mathbb{S}^d}|\varphi|^2 d\mu\geq\left(\int_{\mathbb{S}^d}|\varphi|^q d\mu\right)^{2/q}.$$

Here $d\mu$ is the normalized Lebesque measure on $\mathbb{S}^d$:

$$d\mu=\frac{d\sigma}{\sigma_d}=\frac{d\sigma}{\frac{2\pi^{\frac{d+1}{2}}}{\Gamma(\frac{d+1}{2})}},$$

so that $\mu(\mathbb{S}^d)=1$ (the gradient is with respect to the natural metric).
Next, $q\in[2,\infty)$ for $d=1,2$, and $q\in[2,2d/(d-2)]$ for $d\geq 3$.
The remarkable fact is that $(q-2)/d$ is sharp for *all* admissible q.

The inequality

$$\frac{q-2}{d}\int_{\mathbb{S}^d}|\nabla\varphi|^2 d\mu+\int_{\mathbb{S}^d}|\varphi|^2 d\mu\ge\left(\int_{\mathbb{S}^d}|\varphi|^q d\mu\right)^{2/q}.$$

clearly degenerates and turns into the equality on constants. The fact that the constant $(q-2)/d$ is sharp is verified by means of the minimizing sequence $\varphi_\varepsilon(s)=1+\varepsilon v(s)$ as $\varepsilon\to0$, where $v(s)$ is the eigenfunction of the Laplacian on $\mathbb{S}^d$, corresponding to the first positive eigenvalue d.

One cannot get a multiplicative inequality out of it.

Main result for $\mathbb{S}^2$

Theorem

Let a system of zero-mean functions $\{\varphi_j\}_{j=1}^n \in \dot H^1(\mathbb{S}^2)$ *be orthonormal wrt the scalar product*

$$m^2(\varphi_i, \varphi_j) + (\nabla\varphi_i, \nabla\varphi_j) = \delta_{ij}.$$

Then for $1 \le p < \infty$ *the function*

$$\rho(x) := \sum_{j=1}^n |\varphi_j(x)|^2$$

satisfies

$$\|\rho\|_{L^p} \le \mathrm{B}_p m^{-2/p} n^{1/p}, \quad \textit{where} \quad \mathrm{B}_p \le \left(\frac{p-1}{4\pi}\right)^{(p-1)/p}.$$

Corollary

For $\varphi \in \dot{H}^1(\mathbb{S}^2)$ it holds:

$$\|\varphi\|_{L^q(\mathbb{S}^2)} \le \left(\frac{1}{4\pi}\right)^{\frac{q-2}{2q}} \left(\frac{q}{2}\right)^{1/2} \|\varphi\|^{2/q}\|\nabla\varphi\|^{1-2/q}, \qquad q \ge 2.$$

If $n = 1$ the inequality in the theorem becomes

$$\|\varphi\|^2_{L^{2p}} \le \mathrm{B}_p \left(m^{2-2/p}\|\varphi\|^2 + m^{-2/p}\|\nabla\varphi\|^2\right).$$

Optimization wrt m gives

$$\|\varphi\|^2_{L^{2p}} \le \mathrm{B}_p \frac{p}{(p-1)^{(p-1)/p}} \|\varphi\|^{2/p}\|\nabla\varphi\|^{2-2/p}$$
$$= \left(\frac{1}{4\pi}\right)^{(p-1)/p} p\,\|\varphi\|^{2/p}\|\nabla\varphi\|^{2-2/p},$$

which is what we need if we set $2p =: q$.

Vector inequality on $\mathbb{S}^2$

Corollary

Let $\Omega \subseteq \mathbb{S}^2$ *be an arbirtary domain. Let* $u \in \mathbf{H}^1_0(\Omega)$ *and* $\operatorname{div} u = 0$*. Then:*

$$\|u\|_{L^q(\Omega)} \leq \left(\frac{1}{4\pi}\right)^{\frac{q-2}{2q}} \left(\frac{q}{2}\right)^{1/2} \|u\|^{2/q} \|\operatorname{curl} u\|^{1-2/q}, \qquad q \geq 2.$$

Key estimate for the series

Lemma

For $p > 1$ and $m \geq 0$ it holds

$$J_p(m) := m^{2(p-1)}(p-1)\sum_{n=1}^{\infty}\frac{2n+1}{(m^2+n(n+1))^p} < 1.$$

Euler–Maclaurin summation formula.
Computer shows that for each p tested there is monotonicity. If monotonicity is there, then there is nothing more to prove, since

$$J_p(\infty) = 1.$$

Proof

$\Delta = \operatorname{div} \nabla$ on $\mathbb{S}^2$:

$$-\Delta Y_n^k = n(n+1) Y_n^k, \quad k = 1, \dots, 2n+1, \quad n = 0, 1, 2, \dots .$$

The Y_n^k's are orthonormal eigenfunctions and each eigenvalue $\Lambda_n := n(n+1)$ has myltiplicity $2n+1$.
The important identity: $s \in \mathbb{S}^2$

$$\sum_{k=1}^{2n+1} Y_n^k(s)^2 = \frac{2n+1}{4\pi}.$$

Two operators

$$\mathbb{H} = V^{1/2}(m^2 - \Delta)^{-1/2}\Pi, \quad \mathbb{H}^* = \Pi(m^2 - \Delta)^{-1/2} V^{1/2},$$

where $V \in L^p$, $p > 1$, $V(s) \geq 0$.

Π is the projection on zero-mean functions:

$$\Pi\varphi = \varphi - \frac{1}{4\pi}\int_{\mathbb{S}^2}\varphi(s)d\sigma.$$

Then $\mathbf{K} = \mathbb{H}^*\mathbb{H}$ is a compact self-adjoint operator in $L^2(\mathbb{S}^2)$ and for $r = p' = p/(p-1) \in (1,\infty)$ it holds

$$\begin{aligned}\operatorname{Tr}\mathbf{K}^r &= \operatorname{Tr}\left(\Pi(m^2-\Delta)^{-1/2}V(m^2-\Delta)^{-1/2}\Pi\right)^r\\ &\le \operatorname{Tr}\left(\Pi(m^2-\Delta)^{-r/2}V^r(m^2-\Delta)^{-r/2}\Pi\right)\\ &= \operatorname{Tr}\left(V^r(m^2-\Delta)^{-r}\Pi\right),\end{aligned}$$

where we used Araki–Lieb–Thirring inequality for traces:

$$\operatorname{Tr}(BA^2B)^p \le \operatorname{Tr}(B^pA^{2p}B^p), \quad p \ge 1,$$

the cyclicity property of the trace, and the fact that the projection Π commutes with the Laplacian. Using the above identity we obtain

$$\operatorname{Tr} \mathbf{K}^r \leq \operatorname{Tr}\left(V^r(m^2-\Delta)^{-r}\Pi\right)$$

$$= \int_{\mathbb{S}^2} V(s)^r \sum_{n=1}^{\infty} \sum_{k=1}^{2n+1} \frac{1}{(m^2+n(n+1))^r} Y_n^k(s)^2 d\sigma$$

$$= \frac{1}{4\pi} \sum_{n=1}^{\infty} \frac{2n+1}{\left(m^2+n(n+1)\right)^r} \int_{\mathbb{S}^2} V(s)^r d\sigma \leq \frac{1}{4\pi} \frac{m^{-2(r-1)}}{r-1} \|V\|_{L^r}^r.$$

We can now proceed as in (Lieb 1983). Recall that $\mathbb{H} = V^{1/2}(m^2-\Delta)^{-1/2}\Pi$. Therefore

$$\int_{\mathbb{S}^2} \rho(s)V(s)d\sigma = \int_{\mathbb{S}^2} \sum_{j=1}^{n} |\varphi_j(s)|^2 V(s)d\sigma = \sum_{i=1}^{n} \|\mathbb{H}\psi_i\|_{L^2}^2,$$

where

$$\psi_j = (m^2-\Delta)^{1/2}\varphi_j, \quad j=1,\dots,n.$$

The ψ_j's are orthonormal in L^2:

$$(\psi_i,\psi_j) = (\varphi_i,(m^2-\Delta)\varphi_j) = m^2(\varphi_i,\varphi_j) + (\nabla\varphi_i,\nabla\varphi_j) = \delta_{ij},$$

Variational principle gives that

$$\sum_{i=1}^{n} \|\mathbb{H}\psi_i\|_{L^2}^2 = \sum_{i=1}^{n} (\mathbf{K}\psi_i, \psi_i) \le \sum_{i=1}^{n} \lambda_i,$$

where $\lambda_i > 0$ are the eigenvalues of $\mathbf{K}$. Therefore

$$\int_{\mathbb{S}^2} \rho(s)V(s)d\sigma \le \sum_{i=1}^{n} \lambda_i \le n^{1/p} \left(\mathrm{Tr}\, K^r\right)^{1/r}$$

$$\le n^{1/p} \left(\frac{p-1}{4\pi m^{2/(p-1)}}\right)^{(p-1)/p} \|V\|_{L^{p/(p-1)}}$$

$$= n^{1/p} m^{-2/p} \left(\frac{p-1}{4\pi}\right)^{(p-1)/p} \|V\|_{L^{p/(p-1)}}.$$

Finally, setting $V(x) = \rho(x)^{p-1}$ we obtain

$$\|\rho\|_{L^p} \le \mathrm{B}_p m^{-2/p} n^{1/p}, \quad \text{where} \quad \mathrm{B}_p \le \left(\frac{p-1}{4\pi}\right)^{(p-1)/p}.$$

Conjecture

$$c_q \le \left(\frac{1}{4\pi}\right)^{\frac{q-2}{2q}} \left(\frac{q}{2}\right)^{1/2} \sim \sqrt{q}\sqrt{\frac{1}{8\pi}}$$

1) Is it true that

$$\lim_{q\to\infty} \frac{c_q}{\sqrt{q}} = \sqrt{\frac{1}{8\pi}}.$$

2) Is there an extremal function on the torus and on the sphere?

Two main estimates

$$I_p(m) := \frac{(p-1)m^{2(p-1)}}{\pi} \sum_{n\in\mathbb{Z}_0^2} \frac{1}{(m^2+|n|^2)^p} < 1 \quad \text{on} \quad \mathbb{T}^2$$

$$J_p(m) := m^{2(p-1)}(p-1)\sum_{n=1}^{\infty} \frac{2n+1}{\bigl(m^2+n(n+1)\bigr)^p} < 1 \quad \text{on} \quad \mathbb{S}^2$$

They model the equality in $\mathbb{R}^2$

$$\frac{(p-1)m^{2(p-1)}}{\pi}\int_{\mathbb{R}^2} \frac{dx}{(m^2+|x|^2)^p} = 1$$

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Thank you for your attention