Inertial manifolds I: classical theory

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3. Finite dimensional reduction via inertial manifolds
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Motivation

Dissipative PDEs

e.g. Navier-Stokes equations:

$\partial_t u + (u, \nabla)u + \nabla p = \nu \Delta u + g,$

$\text{div } u = 0.$

Phase space:

$u \in H = L^2(\Omega) \cap \{\text{div } u = 0\},$

H is infinite dimensional

Solution semigroup:

$S(t) : H \to H$

Dissipative estimate:

$$\|S(t)u_0\|_H \leq Ce^{-\gamma t}\|u_0\|_H + C\|g\|_H.$$

Finite-dim reduction $\Longrightarrow$

- Order parameters
- Attractors
- Inertial Manifolds

Apply for turbulence $\Longleftarrow$

Classical Dynamics

System of ODEs:

$\frac{d}{dt}u(t) = F(u(t))$

Phase space:

$u \in H = \mathbb{R}^N,$

H is finite dimensional

Classical methods:

- Bifurcation theory,
- Lyapunov exponents,
- Hyperbolic theory, $\cdots$

Finite dimensional reduction. Way one: global attractor

Definition 1

A set $\mathcal{A}$ is a global attractor for the semigroup $S(t)$ if the following properties are satisfied:

1. *$\mathcal{A}$ is compact in H;*
2. *$\mathcal{A}$ is strictly invariant: $S(t)\mathcal{A} = \mathcal{A}$ for all $t \geq 0$;*
3. *$\mathcal{A}$ is an attracting set for the semigroup $S(t)$, i.e. for every bounded set $B \subset H$ and every neighbourhood $\mathcal{O}(\mathcal{A})$ there exists time $T = T(B, \mathcal{O})$ such that $S(t)B \subset \mathcal{O}(\mathcal{A})$ for all $t \geq T$.*

Definition 2

Let K be a pre-compact set in H. Then, by the Hausdorff criterion, for any $\varepsilon > 0$, K can be covered by finitely many $\varepsilon-$balls in H. Denote by $N_\varepsilon(K, H)$ the minimal number of such balls. Then the fractal dimension is defined via:

$$\dim_f(K, H) := \limsup_{\varepsilon \to 0} \frac{\log N_\varepsilon(K, H)}{\log \frac{1}{\varepsilon}}.$$

Finite dimensional reduction via attractors

Key result: under natural assumptions on a dissipative system $S(t)$, the global attractor $\mathcal{A}$ exists and has the finite fractal and Hausdorff dimensions.

$$+$$

Mané projection theorem: if $\mathcal{A}$ is a compact set in H of finite fractal dimension and $n \geq 2\dim_f(\mathcal{A}, H) + 1$, then an orthoprojector P_L to a generic plane $L \subset H$ of $\dim = n$ is a homeomorphism on $\mathcal{A}$, i.e. $P_L : \mathcal{A} \leftrightarrow P_L\mathcal{A} \subset L = \mathbb{R}^N$.
Let us consider an abstract parabolic equation

$$\frac{d}{dt}u(t) + Au(t) = F(u(t)). \tag{1}$$

Then for any trajectory $u(t) \in \mathcal{A}$ of equation (1), the function $v(t) := Pu(t)$ solves the following system of ODEs

$$\frac{d}{dt}v(t) + P \circ A \circ P^{-1}v(t) = PF(P^{-1}v(t)) - \tag{2}$$

inertial form of the initial PDE.

Drawbacks

Principal problem: drastic loss of regularity.

- The constructed inertial form only Hölder continuous even if the initial problem is analytic.
- Fractal dimension is not appropriate for constructing more regular solutions.
- More delicate dimensions (like Assoud's one) may be infinite $\Rightarrow$ infinite-dimensional dynamics on the attractor of the finite fractal dimension is possible.

Smaller problems:

- Transient behavior is out of control.
- Not clear how to fix a projection plane L.

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* A. Eden, S. Zelik, and V. Kalantarov, *Counterexamples to regularity of Mané projections in the theory of attractors*, Russ. Math. Surv. 68 (2013), no. 2, 199–226.
* S. Zelik, *Inertial manifolds and finite-dimensional reduction for dissipative pdes*, Proceedings of the Royal Society of Edinburgh: Section A Mathematics 144 (2014), 1245–1327.

Finite dimensional reduction. Way two: Inertial Manifolds

Definition 3

The set $\mathcal{M} \subset H$ is called an **inertial manifold** *(IM) for dynamical system $S(t)$ if the following conditions are satisfied:*

1. *The set $\mathcal{M}$ is invariant with respect to the semigroup $S(t)$, i.e. $S(t)\mathcal{M} = \mathcal{M}$;*
2. *$\mathcal{M}$ is finite-dimensional (at least Lipschitz, but usually $C^{1+\varepsilon}$ for $\varepsilon > 0$);*
3. *The exponential tracking property holds, i.e. there exist positive constants C and γ such that for every $u_0 \in H$ there is $v_0 \in \mathcal{M}$ such that*

$$\|S(t)u_0 - S(t)v_0\|_H \le C_{u_0} e^{-\gamma t}, \; t \ge 0.$$

Abstract functional model

Let us consider the following abstract functional model in a Hilbert space H

$$\partial_t u + Au = F(u), \quad u|_{t=0} = u_0, \tag{3}$$

1. $A : D(A) \to H$ is a linear positive self-adjoint operator with compact inverse.
2. $F : H^\beta \to H$, $0 \le \beta < 2$, $H^\beta = D(A^{\beta/2})$, is a non-linear map such that

$$\|F(u) - F(v)\|_H \le L\|u - v\|_{H^\beta}, \quad u, v \in H^\beta. \tag{4}$$

Very often the base of $\mathcal{M}$ is a spectral subspace of A.

$$Ae_n = \lambda_n e_n, \quad 0 < \lambda_1 \leq \lambda_2 \leq \lambda_3 \leq \cdots \tag{5}$$

$$u = \sum_{n=1}^{\infty} u_n e_n, \quad u_n = (u, e_n). \tag{6}$$

$$P_N : H \to H_+, \ H_+ = span\{e_1, \cdots, e_N\}, \ u_+ := P_N u = \sum_{n=1}^{N} u_n e_n,$$

$$Q_N := Id - P_N : H \to H_-, H_- = span\{e_{N+1}, \cdots\}, u_- := Q_N u.$$

Then $H = H_+ \oplus H_-$ and $u = u_+ + u_-$.

$$\begin{cases} \partial_t u_+ + Au_+ = P_N F(u_+ + u_-), \\ \partial_t u_- + Au_- = Q_N F(u_+ + u_-). \end{cases} \tag{7}$$

Inertial form

In this case IM can be presented as a graph of a Lipschitz continuous function $M : H_+ \to H_-$

$$\mathcal{M} = \{u_+ + M(u_+), u_+ \in P_N H\}.$$

And thus on the IM $u_-(t) = M(u_+(t))$.
Dynamics on IM:

$$\partial_t u_+ + Au_+ = P_N F(u_+ + M(u_+)). \tag{8}$$

System of ODEs on order parameters u_+ = Inertial form of equation (3). $\Rightarrow$ (8) holds up to exponentially decaying transient behavior and captures all nontrivial dynamics of (3) $\Rightarrow$ IM gives a rigorous justification for finite dimensional reduction.

Theorem 1 (Existence of an inertial manifold by spectral gap)

Let the above assumptions on the operator A and the non-linearity F hold and let, in addition, for some $N \in \mathbb{N}$, the following spectral gap condition hold:

$$\frac{\lambda_{N+1} - \lambda_N}{\lambda_{N+1}^{\beta/2} + \lambda_N^{\beta/2}} > L, \tag{9}$$

where L is a Lipschitz constant of the non-linearity F. Then, problem (3) *possesses an N-dimensional inertial manifold $\mathcal{M}$. If the nonlinearity F is smoother, then IM is $C^{1+\varepsilon}$-smooth for some small $\varepsilon > 0$.*

- C. Foias, G. Sell, and R. Temam, *Inertial manifolds for nonlinear evolutionary equations*, J. Differential Equations 73 (2) (1988) 309–353.
- M. Miklavcic, *A sharp condition for existence of an inertial manifold,* J. Dynam. Differential Equations, 3, no. 3, (1991), 437–456.
- A. Romanov, *Sharp estimates for the dimension of inertial manifolds for nonlinear parabolic equations*.Russian Acad. Sci. Izv. Math.,43, no. 1, (1994), 31–47.

Advantages

- Simple inertial form of $C^{1+\varepsilon}$-regularity.
- Control of transient terms

$$\alpha \approx \frac{\lambda_{N+1} + \lambda_N}{2} - \text{ high decay rate}$$

Drawbacks

- Very restrictive spectral gap conditions.
- Still low regularity $C^{1+\varepsilon}$
  (not enough for many purposes: bifurcations, numerical schemes, etc.)

Alternative approaches

- Spatial averaging principle
  - J. Mallet-Paret, G. Sell, *Inertial manifolds for reaction-diffusion equations in higher space dimensions.*, J.Amer. Math. Soc.1(1988), 805–866.
  - S. Zelik, *Inertial manifolds and finite-dimensional reduction for dissipative PDEs*, Proc. Royal Soc. Edin-burgh, 144A, (2014), 1245–1327.

  The scalar reaction-diffusion equation was considered

  $$\partial_t u = \Delta_x u - f(u), \quad u(t)|_{t=0} = u_0 \tag{10}$$

  on a 3D torus and proved the existence of a C^1-smooth inertial manifold under some assumptions on the nonlinearity.
- Transformation of the initial equation to another one, for which the spectral gap condition is satisfied
  - M. Kwak, *Finite dimensional inertial forms for 2D Navier-Stokes equations.* Indiana Univ. Math. J. 41 (1992), 927–982.

  Erroneous proof of the existence of IMs for 2D Navier-Stokes equations.

Reaction-diffusion-advection equations

$$\partial_t u - \partial_x^2 u + u + f(u)\partial_x u + g(u) = 0,\ x \in (-\pi, \pi), \qquad (11)$$

- $u = u(t, x) = (u_1, \cdots, u_m)$;
- $f, g \in C_0^\infty$;
- $A := -\partial_x^2 + 1$ endowed by the proper boundary conditions;
- $H := L^2(-\pi, \pi)$;
- $F(u) = -f(u)\partial_x u - g(u)$, therefore $F(u) : H^1(-\pi, \pi) \to H$ and $\beta = 1$;
- $\lambda_n \sim Cn^2$.

Then spectral gap condition reads

$$\frac{\lambda_{N+1} - \lambda_N}{\lambda_{N+1}^{1/2} + \lambda_N^{1/2}} \sim C\frac{(N+1)^2 - N^2}{N + N + 1} = C > L_f, \qquad (12)$$

where L_f is a Lipschitz constant related to f.

Reaction-diffusion-advection equations

$$\partial_t u - \partial_x^2 u + u + f(u)\partial_x u + g(u) = 0, \ x \in (-\pi, \pi),$$

$$u = u(t, x) = (u_1, \cdots, u_m)$$

Dirichlet BC → *Existence of IM*

Neumann BC → *Existence of IM*

Scalar eq. periodic BC → *Existence of IM*

System of eq. with periodic BC → *Nonexistence of IM*

RDA equations with Dirichlet BC

Let us consider RDA equations with Dirichlet boundary conditions and introduce the following change of variables

$$u(t,x) = a(t,x)v(t,x), \text{ where } a(t,x) \in GL(m). \tag{13}$$

Substituting it into reaction-diffusion-advection equation (11), we obtain

$$\partial_t v - \partial_x^2 v + v = a^{-1}[-f(av)a + 2\partial_x a]\partial_x v + \\ + a^{-1}[\partial_x^2 a - \partial_t a - f(av)\partial_x a]v - a^{-1}g(av). \tag{14}$$

Naive idea:
To fix $a = a(v)$ as a solution of the following ODE:

$$\partial_x a = \frac{1}{2} f(av)a, \quad a\big|_{x=0} = \mathrm{Id}\,. \tag{15}$$

But $\partial_x^2 a = \frac{1}{4} f(av)^2 a + \frac{1}{2} f'(av)(\partial_x av + a\partial_x v)a \Rightarrow$ the same spectral gap condition.

Let us fix a as a solution of the following ODE:

$$\partial_x a = \frac{1}{2} f(P_K(av))a, \quad a\Big|_{x=0} = \mathrm{Id}\,. \tag{16}$$

where P_K is an orthoprojector on the first K eigenvectors of the Laplacian $-\partial_x^2$ on $(-\pi, \pi)$ with Dirichlet boundary conditions and $K \gg 1$.

Theorem 2 (A. Kostianko, S. Zelik)

Let the non-linearities f and g be smooth and have finite supports. Then system (11) *with Dirichlet boundary conditions possesses an IM in the phase space $H_0^1(-\pi, \pi)$ and this manifold is $C^{1+\varepsilon}$-smooth where $\varepsilon > 0$ is small enough.*

- A. Kostianko and S. Zelik, *Inertial manifolds for 1D reaction-diffusion-advection systems. Part I: Dirichlet and Neumann boundary conditions*, Commun. Pure Appl. Anal. 16, no. 6 (2017), 2357-2376.

RDA equations with Neumann BC

The transform $u = av$ doesn't preserve Neumann BC. Indeed,

$$0 = \partial_x u\big|_{x=-\pi,\pi} = \partial_x av\big|_{x=-\pi,\pi} + a\partial_x v\big|_{x=-\pi,\pi}. \tag{17}$$

Alternative way

To reduce Neumann BC to Dirichlet BC. Indeed, let $w = \partial_x u$. Then functions (u, w) solve

$$\begin{cases} \partial_t u + f(u)w + g(u) = \Delta_x u - u, \qquad \partial_x u\big|_{x=-\pi,\pi} = 0, \\ \partial_t w + f'(u)[w, w] + g'(u)w + f(u)\nabla_x w = \Delta_x w - w, \\ \hfill w\big|_{x=-\pi,\pi} = 0. \end{cases} \tag{18}$$

Theorem 3 (A. Kostianko, S. Zelik)

Let the nonlinearities f and g be smooth and have finite supports. Then equation (11) with Neumann BC possesses an IM in the phase space $H^1(-\pi,\pi)$ and this manifold is $C^{1+\varepsilon}$-smooth where $\varepsilon > 0$ is small enough.

Scalar RDA equation with periodic BC

Theorem 4 (A. Kostianko, S. Zelik)

Let the nonlinearities f and g be smooth and have finite supports. Then a scalar equation (11) with periodic BC possesses an IM in the phase space $H^1(-\pi, \pi)$ and this manifold is $C^{1+\varepsilon}$-smooth where $\varepsilon > 0$ is small enough.

- A. Kostianko and S. Zelik, *Inertial manifolds for 1D reaction-diffusion-advection systems. Part II: periodic boundary conditions*, Commun. Pure Appl. Anal. 17, no. 1 (2018), 285-317.

System of RDA equations with periodic BC

Theorem 5 (A. Kostianko, S. Zelik)

There exist $m > 1$ and the nonlinearities $f, g \in C_0^\infty$ such that the associated RDA system (11) with periodic boundary conditions does not possess any finite-dimensional IM containing the global attractor. Moreover, the associated limit dynamics on the attractor is infinite-dimensional and, in particular, contains limit cycles with supra exponential rate of attraction.

The proof is based on the proper counterexample to the Floquet theory for linear equations with time-periodic coefficients. Namely, we have found smooth space-time periodic functions $f(t, x)$ and $g(t, x)$ such that all solutions of

$$\partial_t u - \partial_x^2 u + u + f(t,x)\partial_x u + g(t,x)u = 0, \qquad x \in (-\pi, \pi). \quad (19)$$

decay faster than exponentially as $t \to \infty$ (actually, the decay rate is like $e^{-\kappa t^3}$ for some positive κ).