ON THE STRONGLY DAMPED WAVE EQUATION WITH MEMORY

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Abstract. A semilinear strongly damped wave equation with memory is considered in the past history framework. The existence of global attractors of optimal regularity is established, both for critical and supercritical nonlinearities, under a necessary and sufficient condition on the memory kernel.

1. Introduction

Let $\Omega \subset \mathbb{R}^3$ be a bounded domain with smooth boundary $\partial\Omega$. For $t > 0$, we consider the semilinear strongly damped wave equation with memory

$$\partial_{tt} u - \alpha\Delta\partial_t u - \beta\Delta u + \int_0^\infty \mu(s)\Delta u(t-s)\mathrm{d}s + \varphi(u) = f, \tag{1.1}$$

where α, β are (strictly) positive constants, and the *memory kernel* μ is a nonnegative decreasing function on $\mathbb{R}^+ = (0, \infty)$ satisfying the integrability condition

$$\int_0^\infty \mu(s)\mathrm{d}s < \beta.$$

The equation is supplemented with the initial and boundary conditions

$$\begin{cases} u(0) = u_0, \\ \partial_t u(0) = u_1, \\ u(t)_{|t<0} = g(t), \\ u(t)_{|\partial\Omega} = 0, \end{cases} \tag{1.2}$$

where u_0, u_1 and g (the *past history* of u) are given data.

Equation (1.1) originates from a scalar homogenized model in linear viscoelasticity, when a nonlinear displacement-dependent external force is applied. Indeed, the homogenization process of a composite material, whose microscopic mechanical behavior is described by a viscoelastic stress-strain relation of Kelvin-Voigt type, leads to a macroscopic stress-strain relation containing a time convolution term accounting for memory effects (see [8, 20] and references therein).

The case $\mu \equiv 0$ corresponds to to the well-known strongly damped wave equation

$$\partial_{tt} u - \alpha\Delta\partial_t u - \beta\Delta u + \varphi(u) = f, \tag{1.3}$$

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widely studied in the literature (see, e.g., [3, 4, 7, 10, 16, 17, 18, 21, 22, 23, 26, 27]). Well-posedness results and the existence of a global attractor for (strong) solutions to (1.3) have been established in the paper [17] (see also [18]), working in the phase space $[H^2(\Omega)\cap H_0^1(\Omega)]\times H_0^1(\Omega)$, under the assumptions

$$\varphi'(x)\geq -C_1,\qquad x\varphi(x)-\int_0^x\varphi(y)\mathrm{d}y\geq -C_2,\qquad \int_0^x\varphi(y)\mathrm{d}y\geq -C_3,$$

for every $x\in\mathbb{R}$ and some $C_i\geq 0$, but without requiring any growth restriction on the nonlinear term φ. In presence of a nonlinearity of (polynomial) order less than or equal to 5, equation (1.3) is also well-posed in the weak energy phase space $H_0^1(\Omega)\times L^2(\Omega)$, and the related semigroup has a global attractor (see [4, 16, 22] for the critical nonlinearity of order 5). Nonetheless, the optimal regularity of the attractor has been proved only in the recent paper [23], exploiting the peculiar parabolic nature of the equation, as suggested by [17].

The strongly damped wave equation with memory (1.1) has been considered in [2] (see also [12]), for a *subcritical* nonlinearity and assuming the inequality

$$\mu'(s)+\delta\mu(s)\leq 0,\qquad \forall s>0, \tag{1.4}$$

for some $\delta>0$. In that paper, the existence of a global attractor has been proved in the so-called *past history framework*, i.e., in the phase space $H_0^1(\Omega)\times L^2(\Omega)\times L^2_\mu(\mathbb{R}^+,H_0^1(\Omega))$. On the other hand, condition (1.4), commonly employed to obtain asymptotic results for semigroups arising from problems with memory, is rather restrictive. Indeed, let alone that μ is positive and decreasing for physical reasons, (1.4) prevents μ to have flat zones, or even to exhibit horizontal inflection points. A much weaker condition (see [5]), is to require that, for every $\sigma\geq 0$ and (almost) every $s>0$,

$$\mu(s+\sigma)\leq Ke^{-\delta\sigma}\mu(s), \tag{1.5}$$

for some $K\geq 1$ and $\delta>0$, which is equivalent to (1.4) if $K=1$. Nonetheless, when $K>1$, the gap between (1.4) and (1.5) is quite relevant. For instance, any decreasing kernel eventually vanishing, such as a one-step function, fulfills (1.5). This would allow to include in the analysis an equation with delay of the form

$$\partial_{tt}u-\alpha\Delta\partial_t u-\beta\Delta u+\int_0^1\Delta u(t-s)\mathrm{d}s+\varphi(u)=f.$$

The aim of the present paper is to prove the existence of global attractors of optimal regularity for the semigroup generated by (1.1)-(1.2) in the past history framework, considering critical nonlinearities and replacing (1.4) with the more satisfactory condition (1.5). In fact, analogously to [23], we will be able to treat even supercritical nonlinearities, working in the phase space of strong solutions. Besides, we will show that (1.5) is actually necessary in order to obtain the existence of the attractor.

However, the presence of the convolution integral introduces substantial complications. Consequently, the scheme of [23] cannot be directly reproduced, and a skillful treatment of the memory is required. Indeed, when looking for dissipative estimates, the peculiar structure of this kind of equations has always been thought to lead to the natural multiplications, of hyperbolic flavor, by $\partial_t u$ or $-\Delta\partial_t u$, which produce an automatic cancellation of a term otherwise impossible to handle (cf. [2, 6]). On the other hand, an analogous cancellation does not occur when multiplying the equation by u or $-\Delta u$, which is exactly what is needed to pursue the "parabolic" strategy devised in [23]. Hence, our main effort here is to enucleate the partial parabolicity of the equation, which is a harder task than in [23], due to the further hyperbolicity caused by the memory, and then develop new techniques that allow to control the additional memory terms that pop up under this, apparently unnatural, multiplication. This is perhaps the main novelty of the paper: to the best of our knowledge, in all the previous works dealing with problems with memory, the exploitation of the aforementioned cancellation has always been considered the only possible way to proceed. On the contrary, our alternative approach seems to be apt to tackle other equations as well, whenever regularizing terms are present.

Plan of the paper. In the next Section 2, we introduce the functional setting and the basic assumptions. The main results of the paper are stated in Section 3. Section 4 is devoted to several technical results on equations in memory spaces. The proofs of the main theorems are carried out in Section 5 and Section 6.

2. Notation and Basic Assumptions

Along this paper, C and $\mathcal{Q}$ will stand for a generic positive constant and a generic positive increasing function, respectively, depending only on the structural quantities of the problem under consideration. The symbol Λ will be used to denote certain energy functionals occurring in the proofs.

Notation. We indicate by $\|\cdot\|$ and $\langle\cdot,\cdot\rangle$ the standard norm and inner product on $L^2(\Omega)$. Naming $A=-\Delta$ with Dirichlet boundary conditions, we consider, for $\ell\in\mathbb{R}$, the scale of Hilbert spaces

$$H_\ell=\mathcal{D}(A^{\ell/2}),\qquad \langle u,v\rangle_\ell=\langle A^{\ell/2}u,A^{\ell/2}v\rangle.$$

In particular, $H_0=L^2(\Omega)$, $H_1=H_0^1(\Omega)$ and $H_2=H^2(\Omega)\cap H_0^1(\Omega)$. Recall that, for every $\ell\geq 0$ and every $u\in H_\ell$,

$$\|u\|\leq {\lambda_1}^{-\ell/2}\|u\|_\ell,$$

where $\lambda_1>0$ is the first eigenvalue of A. Then, we define the L^2-weighted spaces (the so-called *memory spaces*)

$$\mathcal{M}_\ell=L^2_\mu(\mathbb{R}^+,H_\ell),\qquad \langle\eta,\psi\rangle_{\ell,\mu}=\int_0^\infty\mu(s)\langle\eta(s),\psi(s)\rangle_\ell\,\mathrm{d}s,$$

along with the infinitesimal generator of the right-translation semigroup on $\mathcal{M}_\ell$, that is, the linear operator $T\eta=-D\eta$ with domain

$$\mathcal{D}(T)=\{\eta\in\mathcal{M}_\ell: D\eta\in\mathcal{M}_\ell,\ \eta(0)=0\},$$

where D stands for the distributional derivative, and $\eta(0)=\lim_{s\to 0}\eta(s)$ in H_ℓ. We will also make use of the H^1-weighted spaces

$$\mathcal{M}^1_\ell=H^1_\mu(\mathbb{R}^+,H_\ell),\qquad \langle\eta,\psi\rangle_{\ell,\mu;1}=\int_0^\infty\mu(s)\big[\langle\eta(s),\psi(s)\rangle_\ell+\langle D\eta(s),D\psi(s)\rangle_\ell\big]\mathrm{d}s.$$

Finally, we introduce the product Hilbert spaces

$$\mathcal{H}=H_1\times H_0\times\mathcal{M}_1\qquad\text{and}\qquad\mathcal{V}=H_2\times H_0\times\mathcal{M}_2.$$

The following general assumptions on μ, f and φ are understood to hold in all the statements of the paper.

Assumptions on $\boldsymbol{\mu}$. Let $\mu : \mathbb{R}^+ \to [0,\infty)$ be a decreasing function such that

$$\kappa = \int_0^\infty \mu(s)\mathrm{d}s < \infty.$$

We assume that there exists $\Theta > 0$ such that, for every $s \in \mathbb{R}^+$,

$$k(s) = \int_s^\infty \mu(\sigma)\mathrm{d}\sigma \leq \Theta\mu(s), \tag{2.1}$$

We allow μ to have jumps at $s = s_n$, where $\{s_n\}$ is a strictly increasing sequence, with $s_0 = 0$, either finite (possibly reduced to s_0 only) or converging to $s_\infty \in (0,\infty]$, whereas we require the absolute continuity of μ on each interval (s_{n-1}, s_n) and on the interval (s_∞, ∞), if defined. In particular, the derivative μ' exists (nonpositive) almost everywhere.

As shown in [9], condition (2.1) is equivalent to (1.5). As a consequence,

$$k(s) \leq Ce^{-\delta s}. \tag{2.2}$$

Assumptions on $\boldsymbol{f}$ and $\boldsymbol{\varphi}$. Let $f \in H_0$ be independent of time and $\varphi \in C^1(\mathbb{R})$, with $\varphi(0) = 0$, be such that

$$\liminf_{|x|\to\infty} \varphi'(x) > -\lambda_1. \tag{2.3}$$

In particular, (2.3) implies that

$$\varphi'(x) \geq -\omega. \tag{2.4}$$

for some $\omega \geq \lambda_1$. No growth restrictions on φ are required.

Setting

$$\Phi(u) = \langle \widetilde{\varphi}(u), 1 \rangle,$$

with $\widetilde{\varphi}(x) = \int_0^x \varphi(y)\mathrm{d}y$, the following inequalities are easily verified:

$$\Phi(u) \geq -\frac{\vartheta}{2}\|u\|^2 - C, \tag{2.5}$$

$$\langle \varphi(u), u \rangle \geq \Phi(u) - \frac{\vartheta}{2}\|u\|^2 - C \geq -\vartheta\|u\|^2 - C, \tag{2.6}$$

for some $\vartheta < \lambda_1$.

Remark 2.1. Throughout this work, we will perform several formal estimates, which hold true when the functions involved are regular enough (in particular, functions in memory spaces should belong to $\mathcal{D}(T)$). As usual, these estimates can be made rigorous working in a suitable approximation scheme (see [11, 24]). Besides, we will use many times the standard Hölder and Young inequalities and the Sobolev embeddings without explicit mention.

3. The Main Results

We introduce the past history variable [6]

$$\eta^t(s) = u(t) - u(t-s), \qquad s \in \mathbb{R}^+,$$

which (formally) satisfies the differential equation

$$\partial_t \eta^t(s) = -\partial_s \eta^t(s) + \partial_t u(t)$$

with the initial condition

$$\eta^0(s) = \eta_0(s),$$

with $\eta_0(s) = u_0 - g(-s)$. Taking for simplicity $\alpha = 1$ and $\beta - \int_0^\infty \mu(s)\mathrm{d}s = 1$, we translate (1.1)-(1.2) into the Cauchy problem in the history space framework

$$\begin{cases} \partial_{tt}u + A\partial_t u + Au + \varphi(u) + \displaystyle\int_0^\infty \mu(s)A\eta(s)\mathrm{d}s = f, \\ \partial_t \eta = T\eta + \partial_t u, \\ (u(0), \partial_t u(0), \eta^0) = z, \end{cases} \tag{3.1}$$

having set

$$z = (u_0, u_1, \eta_0).$$

In fact, under suitable assumptions, there is a complete equivalence between (1.1)-(1.2) and (3.1) (see [13]).

Then, we have

Theorem 3.1. *Problem* (3.1) *generates a strongly continuous semigroup* $S(t)$ *on the phase space* $\mathcal{V}$.

Theorem 3.2. *Assuming in addition the (critical) growth condition*

$$|\varphi'(x)| \leq C(1 + |x|^4), \tag{3.2}$$

problem (3.1) *generates a strongly continuous semigroup* $S(t)$ *on the phase space* $\mathcal{H}$.

We denote the corresponding (twice the) energies by

$$\mathcal{E}_{\mathcal{V}}(t) = \|S(t)z\|_{\mathcal{V}}^2, \qquad \mathcal{E}_{\mathcal{H}}(t) = \|S(t)z\|_{\mathcal{H}}^2.$$

We omit the proofs of Theorem 3.1 and Theorem 3.2, which can be obtained in a standard way, by means of a Galerkin approximation scheme (cf. [11, 24] to deal with the memory part), using the dissipative estimates proved later in this work. Again, we stress that no growth restrictions on φ are required in order to prove the existence of the semigroup in $\mathcal{V}$, whereas in $\mathcal{H}$, for nonlinearities exceeding the critical exponent 5, we lose uniqueness.

Our main results, whose proofs are postponed in the next sections, read as follow.

Theorem 3.3. *The semigroup* $S(t)$ *on* $\mathcal{V}$ *possesses a connected global attractor* $\mathcal{A}$ *which is bounded in* $H_2 \times H_2 \times \mathcal{M}_3$. *Besides, the third component of* $\mathcal{A}$ *is included in* $\mathcal{D}(T)$, *bounded in* $\mathcal{M}_2^1$ *and pointwise bounded in* H_3.

Remark 3.4. In fact, as will be clear from the proof, the first component of the set $\mathcal{A} - (A^{-1}f, 0, 0)$ is bounded in H_3. Thus, if $f \in H_1$, then $\mathcal{A}$ is bounded in $H_3 \times H_2 \times \mathcal{M}_3$.

Recall that the global attractor is the unique compact subset of the phase space which is fully invariant for $S(t)$ and attracts all bounded subsets with respect to the Hausdorff semidistance (see [1, 14, 15, 19, 25] for more details on this theory).

The regularity of $\mathcal{A}$, which is optimal within our hypotheses, can be improved up to where the regularity of φ and f permit.

Corollary 3.5. *If $\varphi \in C^\infty(\mathbb{R})$ and $f \in C^\infty(\overline{\Omega})$, then*

$$\mathcal{A} \subset C^\infty(\overline{\Omega}) \times C^\infty(\overline{\Omega}) \times C^\infty([0,\infty), C^\infty(\overline{\Omega})).$$

Theorem 3.6. *If the growth condition* (3.2) *holds, the semigroup $S(t)$ on $\mathcal{H}$ possesses a global attractor which coincides with $\mathcal{A}$.*

Remark 3.7. Assumption (2.1) is sharp in order to obtain the existence of the global attractor. To prove this fact, assume that the conclusion of Theorem 3.6 holds *without* requiring (2.1) (which, in any case, is not needed to prove the existence of the semigroup). When $\varphi = 0$ and $f = 0$, the system is linear homogeneous, and $\mathcal{A}$ reduces to $\{0\}$. In that case, Theorem 3.6 translates into the exponential stability of the (linear) semigroup $S(t)$ on $\mathcal{H}$. In particular, for initial data $z = (0, 0, \eta_0)$,

$$\|u(t)\|_1^2 + \|\eta^t\|_{1,\mu}^2 \leq M\|\eta_0\|_{1,\mu}^2 e^{-\delta t},$$

for some $\delta > 0$ and some $M \geq 1$. This implies that the right-translation semigroup $\Sigma(t)$ on $\mathcal{M}_1$, defined as

$$[\Sigma(t)\eta_0](s) = \begin{cases} 0 & 0 < s \leq t, \\ \eta_0(s-t) & s > t, \end{cases}$$

is exponentially stable of exponential rate at least δ. Indeed, exploiting the representation formula (4.1) for η (see the next section), we have

$$\begin{aligned} \|\eta^t\|_{1,\mu}^2 &\geq \int_t^\infty \mu(s)\|\eta_0(s-t) + u(t)\|_1^2 \mathrm{d}s \\ &\geq \frac{1}{2}\int_t^\infty \mu(s)\|\eta_0(s-t)\|_1^2 \mathrm{d}s - M\kappa e^{-\delta t}\|\eta_0\|_{1,\mu}^2 \\ &= \frac{1}{2}\|\Sigma(t)\eta_0\|_{\mathcal{M}_1}^2 - M\kappa e^{-\delta t}\|\eta_0\|_{1,\mu}^2. \end{aligned}$$

Hence, setting $K = 2M(1+\kappa)$, we obtain the inequality

$$\|\Sigma(t)\eta_0\|_{1,\mu}^2 - Ke^{-\delta t}\|\eta_0\|_{1,\mu}^2 = \int_0^\infty \big[\mu(t+s) - Ke^{-\delta t}\mu(s)\big]\|\eta_0(s)\|_1^2 \mathrm{d}s \leq 0.$$

Using an argument devised in [5], for any fixed t, let

$$\mathcal{O}_t = \{s \in \mathbb{R}^+ : \mu(t+s) - Ke^{-\delta t}\mu(s) > 0\}.$$

Choosing $\eta_0(s) = \chi_{\mathcal{O}_t}(s)u_0$, with $u_0 \in H_1$ such that $\|u_0\|_1 = 1$, we conclude that

$$\int_{\mathcal{O}_t} \big[\mu(t+s) - Ke^{-\delta t}\mu(s)\big]\mathrm{d}s = 0,$$

which yields (1.5) and, in turn, its equivalent formulation (2.1).

4. Some Results on Equations with Memory

In this section, we collect several technical results concerning equations in memory spaces, which is really all we need to adapt the arguments devised in [23] for the strongly damped wave equation to the case with memory.

For every $t_0 > 0$, let
$$q \in H^1([0,t_0], H_\ell),$$
and let $\psi^t(s)$ be the unique solution to the Cauchy problem in $\mathcal{M}_\ell$
$$\begin{cases} \partial_t \psi^t = T\psi^t + \partial_t q(t), \\ \psi^0 = \psi_0, \end{cases}$$
for some $\psi_0 \in \mathcal{M}_\ell$. Then, ψ^t has the explicit representation formula (see [24])
$$\psi^t(s) = \begin{cases} q(t) - q(t-s) & 0 < s \le t, \\ \psi_0(s-t) + q(t) - q(0) & s > t. \end{cases} \tag{4.1}$$

Lemma 4.1. *We have the inequality*
$$\|\psi^t\|^2_{\ell,\mu} \le C\|q\|^2_{L^\infty([0,t],H_\ell)} + C\|\psi_0\|^2_{\ell,\mu} e^{-\delta t}.$$
where δ is given by (1.5)*. Furthermore, if*
$$\|q(t)\|^2_\ell \le Me^{-\nu t},$$
for some $\nu \le \delta$ and some $M \ge 0$, then
$$\|\psi^t\|^2_{\ell,\mu} \le C(M + \|\psi_0\|^2_{\ell,\mu})e^{-\nu t}$$

Proof. Due to (1.5),
$$\int_t^\infty \mu(s)\|\psi_0(s-t)\|^2_\ell \mathrm{d}s = \int_0^\infty \mu(s+t)\|\psi_0(s)\|^2_\ell \mathrm{d}s \le Ke^{-\delta t}\|\psi_0\|^2_{\ell,\mu}.$$
Thus, on account of (4.1), and exploiting (2.2), we obtain
$$\begin{aligned} \|\psi^t\|^2_{\ell,\mu} &= \int_0^t \mu(s)\|q(t) - q(t-s)\|^2_\ell \mathrm{d}s + \int_t^\infty \mu(s)\|\psi_0(s-t) + q(t) - q(0)\|^2_\ell \mathrm{d}s \\ &\le 2\kappa\|q(t)\|^2_\ell + 2\int_0^t \mu(s)\|q(t-s)\|^2_\ell \mathrm{d}s + 2Ke^{-\delta t}\|\psi_0\|^2_{\ell,\mu} + Ce^{-\delta t}\|q(t) - q(0)\|^2_\ell. \end{aligned}$$
This proves the first statement. The second one follows from the estimate
$$\int_0^t \mu(s)e^{-\nu(t-s)}\mathrm{d}s \le Ce^{-\nu t},$$
which is a direct consequence of (1.5). ☐

Lemma 4.2. *Assume that $\psi_0 = 0$. Then,*
$$\sup_{s>0} \|\psi^t(s)\|_\ell \le C\|q\|_{L^\infty([0,t],H_\ell)}.$$
If in addition $\partial_t q \in L^\infty([0,t], H_\ell)$, we have the further estimate
$$\|\psi^t\|_{\ell,\mu;1} \le C\|q\|_{L^\infty([0,t],H_\ell)} + C\|\partial_t q\|_{L^\infty([0,t],H_\ell)}.$$

Proof. Use (4.1) and apply Lemma 4.1, noting that

$$D\psi^t(s) = \begin{cases} \partial_t q(t-s) & 0 < s \le t, \\ 0 & s > t, \end{cases}$$

as $\psi_0 = 0$. □

For further use, we introduce the (positive) functionals

$$\Psi_\ell[\psi^t] = \int_0^\infty k(s)\|\psi^t(s) - q(t)\|_\ell^2 \mathrm{d}s,$$

with $k(s) = \int_s^\infty \mu(\sigma)\mathrm{d}\sigma$, and

$$\Gamma_\ell[\psi^t] = -\int_0^\infty \mu'(s)\|\psi^t(s)\|_\ell^2 \mathrm{d}s + \sum_n [\mu(s_n^-) - \mu(s_n^+)]\|\psi^t(s_n)\|_\ell^2.$$

The above sum, accounting for the jumps of μ at $s = s_n$, includes the value $n = \infty$ if $s_\infty < \infty$. By direct calculations, we have the equalities

$$\langle T\psi^t, \psi^t\rangle_{\ell,\mu} = -\frac{1}{2}\Gamma_\ell[\psi^t] \le 0, \tag{4.2}$$

and

$$\frac{\mathrm{d}}{\mathrm{d}t}\Psi_\ell[\psi^t] + \|\psi^t\|_{\ell,\mu}^2 = 2\int_0^\infty \mu(s)\langle \psi^t(s), q(t)\rangle_\ell \, \mathrm{d}s. \tag{4.3}$$

Moreover, from (2.1), we learn that

$$\Psi_\ell[\psi^t] \le C\big(\|q(t)\|_\ell^2 + \|\psi^t\|_{\ell,\mu}^2\big). \tag{4.4}$$

Lemma 4.3. *Let $\ell = 1$, and let $b \in H_1$ be a fixed vector. Then, for every $\varepsilon > 0$ and every $t > 0$,*

$$|\langle T\psi^t, b\rangle_{1,\mu}| \le \varepsilon(1+\mu(t))\|b\|_1^2 + \frac{Q}{\varepsilon}\Big[\mu(t)\|q\|_{L^\infty([0,t],H_1)}^2 + \Gamma_1[\psi^t] + \int_0^t \mu(s)\|\partial_t q(t-s)\|_1^2 \mathrm{d}s\Big],$$

for some $Q \ge 0$.

Proof. For every $t > 0$,

$$\langle T\psi^t, b\rangle_{1,\mu} = \int_0^t \mu(s)\langle T\psi^t(s), b\rangle_1 \mathrm{d}s - \int_t^\infty \mu(s)\frac{\mathrm{d}}{\mathrm{d}s}\langle \psi^t(s), b\rangle_1 \mathrm{d}s.$$

When $s \le t$, from (4.1) we have that

$$T\psi^t(s) = -\partial_t q(t-s).$$

Hence, the first term in the rhs is controlled as

$$\begin{aligned} \Big|\int_0^t \mu(s)\langle T\psi^t, b\rangle_1 \mathrm{d}s\Big| &\le \|b\|_1 \int_0^t \mu(s)\|\partial_t q(t-s)\|_1 \mathrm{d}s \\ &\le \varepsilon\|b\|_1^2 + \frac{C}{\varepsilon}\int_0^t \mu(s)\|\partial_t q(t-s)\|_1^2 \mathrm{d}s. \end{aligned} \tag{4.5}$$

As far as the second term is concerned, recalling that (cf. [13])

$$\lim_{s\to\infty} \mu(s)\|\psi^t(s)\|_1 = 0,$$

integrating by parts, we are led to the equality

$$\begin{aligned}
&-\int_t^\infty \mu(s)\frac{\mathrm{d}}{\mathrm{d}s}\langle\psi^t(s), b\rangle_1 \mathrm{d}s \\
&= \mu(t^+)\langle\psi^t(t), b\rangle_1 - \sum_{s_n>t}[\mu(s_n^-) - \mu(s_n^+)]\langle\psi^t(s_n), b\rangle_1 + \int_t^\infty \mu'(s)\langle\psi^t(s), b\rangle_1 \mathrm{d}s.
\end{aligned}$$

Appealing again to (4.1),

$$|\mu(t^+)\langle\psi^t(t), b\rangle_1| \le \mu(t)\|q(t) - q(0)\|_1\|b\|_1 \le \frac{\varepsilon}{3}\mu(t)\|b\|_1^2 + \frac{C}{\varepsilon}\mu(t)\|q\|^2_{L^\infty([0,t],H_1)}.$$

Since

$$0 \le \sum_{s_n>t}[\mu(s_n^-) - \mu(s_n^+)] \le \mu(t),$$

we obtain

$$\begin{aligned}
\Big|\sum_{s_n>t}[\mu(s_n^-) - \mu(s_n^+)]\langle\psi^t(s_n), b\rangle_1\Big| &\le \sum_{s_n>t}[\mu(s_n^-) - \mu(s_n^+)]\|\psi^t(s_n)\|_1\|b\|_1 \\
&\le \frac{\varepsilon}{3}\mu(t)\|b\|_1^2 + \frac{C}{\varepsilon}\sum_n[\mu(s_n^-) - \mu(s_n^+)]\|\psi^t(s_n)\|_1^2.
\end{aligned}$$

Finally,

$$\begin{aligned}
\Big|\int_t^\infty \mu'(s)\langle\psi^t(s), b\rangle_1 \mathrm{d}s\Big| &\le -\frac{\varepsilon}{3}\|b\|_1^2\int_t^\infty \mu'(s)\mathrm{d}s - \frac{C}{\varepsilon}\int_t^\infty \mu'(s)\|\psi^t(s)\|_1^2 \mathrm{d}s \\
&\le \frac{\varepsilon}{3}\mu(t)\|b\|_1^2 - \frac{C}{\varepsilon}\int_0^\infty \mu'(s)\|\psi^t(s)\|_1^2 \mathrm{d}s.
\end{aligned}$$

In summary,

$$\Big|\int_t^\infty \mu(s)\frac{\mathrm{d}}{\mathrm{d}s}\langle\psi^t(s), b\rangle_1 \mathrm{d}s\Big| \le \varepsilon\mu(t)\|b\|_1^2 + \frac{C}{\varepsilon}\mu(t)\|q\|^2_{L^\infty([0,t],H_1)} + \frac{C}{\varepsilon}\Gamma_1[\psi^t]. \tag{4.6}$$

Collecting (4.5) and (4.6) we are finished. □

5. Proof of Theorem 3.3

We begin to show the existence of a bounded absorbing set for the semigroup $S(t)$ acting on the phase space $\mathcal{V}$.

Theorem 5.1. *The dissipative estimate*

$$\mathcal{E}_{\mathcal{V}}(t) \le \mathcal{Q}(\mathcal{E}_{\mathcal{V}}(0))e^{-\nu t} + C,$$

holds for every $t \ge 0$ and some $\nu > 0$.

In order to prove the result, we need a preliminary lemma.

Lemma 5.2. *We have the estimate*

$$\mathcal{E}_{\mathcal{H}}(t) + \int_t^\infty \|\partial_t u(\tau)\|_1^2 d\tau \le C\big[\mathcal{E}_{\mathcal{H}}(0) + |\Phi(u_0)|\big]e^{-\varepsilon t} + C,$$

for every $t \ge 0$ and some $\varepsilon > 0$.

Proof. For $\varepsilon > 0$ to be determined, we define

$$\Lambda = \mathcal{E}_{\mathcal{H}} + 2\varepsilon\|u\|_1^2 + 2\Phi(u) + 4\varepsilon\langle\partial_t u, u\rangle - 2\langle f, u\rangle + 2\varepsilon\Psi_1[\eta],$$

which, in light of (2.5) and (4.4), satisfies the inequalities

$$\varrho\mathcal{E}_{\mathcal{H}} - C \le \Lambda \le C\mathcal{E}_{\mathcal{H}} + C|\Phi| + C,$$

for some $\varrho > 0$, provided that ε is small enough. Using (3.1), (4.2) and (4.3), we compute the time-derivative of Λ as

$$\frac{\mathrm{d}}{\mathrm{d}t}\Lambda + 2\varepsilon\Lambda + \|\partial_t u\|_1^2 + \Lambda_0 = 0,$$

where we set

$$\begin{aligned}\Lambda_0 &= 2\varepsilon(1-2\varepsilon)\|u\|_1^2 + \|\partial_t u\|_1^2 - 6\varepsilon\|\partial_t u\|^2 - 8\varepsilon^2\langle\partial_t u, u\rangle \\ &\quad + 4\varepsilon\langle\varphi(u), u\rangle - 4\varepsilon\Phi(u) - 4\varepsilon^2\Psi_1[\eta] + \Gamma_1[\eta].\end{aligned}$$

Exploiting (2.6) and (4.4), for ε small enough, we have that

$$\Lambda_0 \ge -\varepsilon\varrho\mathcal{E}_{\mathcal{H}} - \varepsilon C \ge -\varepsilon\Lambda - \varepsilon C.$$

Therefore,

$$\frac{\mathrm{d}}{\mathrm{d}t}\Lambda + \varepsilon\Lambda + \|\partial_t u\|_1^2 \le \varepsilon C,$$

and the Gronwall lemma yields

$$\varrho\mathcal{E}_{\mathcal{H}}(t) - C \le \Lambda(t) \le \Lambda(0)e^{-\varepsilon t} + C \le C\big[\mathcal{E}_{\mathcal{H}}(0) + |\Phi(u_0)|\big]e^{-\varepsilon t} + C.$$

Setting $\varepsilon = 0$ and integrating the above differential inequality on $[t, t_0]$, thanks to the obtained bound on $\Lambda(t)$, the remaining part of the claim follows by letting $t_0 \to \infty$. ☐

Remark 5.3. A closer look into the proof shows that, if $\varphi = 0$ and $f = 0$, we have the exponential decay property (of the related linear semigroup)

$$\mathcal{E}_{\mathcal{H}}(t) \le C\mathcal{E}_{\mathcal{H}}(0)e^{-\varepsilon t}.$$

Proof of Theorem 5.1. Consider the functional

$$\Lambda = \|u\|_2^2 + 2\langle\partial_t u, Au\rangle + \Psi_2[\eta],$$

which, from (4.4), fulfills

$$\nu\Lambda \le \|u\|_2^2 + 2\langle\partial_t u, Au\rangle + \|\eta\|_{2,\mu}^2 \le C\mathcal{E}_{\mathcal{V}},$$

for some $\nu > 0$ small enough. A multiplication of the first equation of (3.1) by Au, together with (2.3) and (4.3), entail

$$\begin{aligned}&\frac{\mathrm{d}}{\mathrm{d}t}\Lambda + \big[\|u\|_2^2 + 2\langle \partial_t u, Au\rangle + \|\eta\|_{2,\mu}^2\big] + \|u\|_2^2 \\ &= 2\|\partial_t u\|_1^2 - 2\langle \varphi'(u)\nabla u, \nabla u\rangle + 2\langle \partial_t u, Au\rangle + 2\langle f, Au\rangle \\ &\leq \|u\|_2^2 + C\big(1 + \|u\|_1^2 + \|\partial_t u\|_1^2\big).\end{aligned}$$

Therefore,

$$\frac{\mathrm{d}}{\mathrm{d}t}\Lambda + \nu\Lambda \leq C\big(1 + \|u\|_1^2 + \|\partial_t u\|_1^2\big).$$

Thus, exploiting Lemma 5.2 and the inequality

$$|\Lambda(0)| + \mathcal{E}_{\mathcal{H}}(0) + |\Phi(u_0)| \leq \mathcal{Q}(\mathcal{E}_{\mathcal{V}}(0)),$$

the Gronwall lemma entails (assuming $\nu \leq \varepsilon$)

$$\Lambda(t) \leq \mathcal{Q}(\mathcal{E}_{\mathcal{V}}(0))e^{-\nu t} + C,$$

thanks to the estimate

$$\int_0^t e^{-\nu(t-\tau)}\|\partial_t u(\tau)\|_1^2 \mathrm{d}\tau \leq e^{-\nu t}\int_0^\infty \|\partial_t u(\tau)\|_1^2\mathrm{d}\tau + \nu\int_0^t e^{-\nu(t-\tau)}\int_\tau^\infty \|\partial_t u(s)\|_1^2 \mathrm{d}s\mathrm{d}\tau,$$

which is obtained integrating by parts. Applying again Lemma 5.2 to control $\|\partial_t u\|^2$, we conclude that

$$\|u(t)\|_2^2 + \|\partial_t u(t)\|^2 \leq \mathcal{Q}(\mathcal{E}_{\mathcal{V}}(0))e^{-\nu t} + C.$$

Assuming, without loss of generality, $\nu \leq \delta$, the remaining estimate for $\|\eta\|_{2,\mu}^2$ follows from the first assertion of Lemma 4.1. $\square$

Thus, Theorem 5.1 provides the existence of a bounded absorbing set $\mathbb{B}_{\mathcal{V}} \subset \mathcal{V}$ for $S(t)$. Following the lines of [23], we now decompose the solution $S(t)z$ with initial data $z \in \mathbb{B}_{\mathcal{V}}$ into the sum

$$S(t)z = (v(t), \partial_t v(t), \xi^t) + (w(t), \partial_t w(t), \zeta^t) + (A^{-1}f, 0, 0),$$

where $(v, \partial_t v, \xi)$ and $(w, \partial_t w, \zeta)$ solve the problems

$$\begin{cases} \partial_{tt}v + A\partial_t v + Av + \displaystyle\int_0^\infty \mu(s)A\xi(s)\mathrm{d}s = 0, \\ \partial_t \xi = T\xi + \partial_t v, \\ (v(0), \partial_t v(0), \xi^0) = z - (A^{-1}f, 0, 0), \end{cases}$$

and

$$\begin{cases} \partial_{tt}w + A\partial_t w + Aw + \varphi(u) + \displaystyle\int_0^\infty \mu(s)A\zeta(s)\mathrm{d}s = 0, \\ \partial_t \zeta = T\zeta + \partial_t w, \\ (w(0), \partial_t w(0), \zeta^0) = 0, \end{cases}$$

Till the end of this section, the generic positive constant C may depend on $\mathbb{B}_{\mathcal{V}}$. Observe that, from Theorem 5.1,

$$\|S(t)z\|_{\mathcal{V}} \leq C. \tag{5.1}$$

Lemma 5.4. *The inequality*

$$\|(v(t),\partial_t v(t),\xi^t)\|_{\mathcal{V}} \leq Ce^{-\nu t}$$

holds for every $t \geq 0$ and some $\nu > 0$.

Proof. Arguing exactly as in the proof of Theorem 5.1 (with v and ξ in place of u and η), and noting that now $\varphi = 0$ and $f = 0$, in view of Remark 5.3, we find the estimate

$$\|v(t)\|_2 + \|\partial_t v(t)\| \leq Ce^{-\nu t},$$

for some $\nu > 0$, and the analogous control for $\|\xi\|_{2,\mu}$ is obtained from the second assertion of Lemma 4.1. □

Lemma 5.5. *The inequality*

$$\|(w(t),\partial_t w(t),\zeta^t)\|_{H_3\times H_2\times \mathcal{M}_3} \leq C.$$

holds for every $t \geq 0$.

Proof. For $\varepsilon > 0$ to be determined, we set

$$\Lambda = (1+\varepsilon)\|w\|_3^2 + \|\partial_t w\|_2^2 + \|\zeta\|_{3,\mu}^2 - 2\varepsilon\langle \partial_t w, Aw\rangle_1 + \varepsilon\Psi_3[\zeta].$$

On account of (4.4), the inequalities

$$\frac{1}{2}\Lambda \leq \|(w,\partial_t w,\zeta)\|_{H_3\times H_2\times \mathcal{M}_3} \leq 2\Lambda$$

hold provided that ε is small enough. Using the equations for $(w,\partial_t w,\zeta)$, along with (4.2) and (4.3), the time-derivative of Λ fulfills the equality

$$\frac{\mathrm{d}}{\mathrm{d}t}\Lambda + 2\varepsilon\|w\|_3^2 + 2(1-\varepsilon)\|\partial_t w\|_3^2 + \varepsilon\|\zeta\|_{3,\mu}^2 = -\Gamma_3[\zeta] - 2\varepsilon\langle\varphi(u),Aw\rangle_1 - 2\langle\varphi(u),A\partial_t w\rangle_1.$$

Since, owing to (5.1) and the assumption $\varphi(0) = 0$, we know that $\varphi(u)$ is uniformly bounded in H_1, we control the rhs by

$$\varepsilon\|w\|_3^2 + \|\partial_t w\|_3^2 + C,$$

Hence, up to further reducing ε, we obtain the differential inequality

$$\frac{\mathrm{d}}{\mathrm{d}t}\Lambda + \varepsilon\Lambda \leq C.$$

The claim follows from an application of the Gronwall lemma. □

As a byproduct, from Lemma 4.2, we have

Lemma 5.6. *The inequality*

$$\|\zeta^t\|_{2,\mu;1} + \sup_{s>0}\|\zeta^t(s)\|_3 \leq C$$

holds for every $t \geq 0$.

We are now in a position to complete the proof of Theorem 3.3. Indeed, Lemma 5.4 and Lemma 5.5 show that the solution $S(t)\mathcal{B}_{\mathcal{V}}$ is (exponentially) attracted by the set $z_0 + \mathcal{K}_{\mathcal{V}}$, where

$$z_0 = (A^{-1}f, 0, 0) \in H_2 \times H_\ell \times \mathcal{M}_\ell$$

for every ℓ, and

$$\mathcal{K}_{\mathcal{V}} = \big\{(\bar u, \bar v, \bar\eta) : \|\bar u\|_3 + \|\bar v\|_2 + \|\bar\eta\|_{3,\mu} + \|\bar\eta\|_{2,\mu;1} + \sup_{s>0} \|\bar\eta(s)\|_3 \le C,\ \bar\eta(0) = 0\big\}.$$

According to [24, Lemma 5.5], $\mathcal{K}_{\mathcal{V}}$ (and so $z_0 + \mathcal{K}_{\mathcal{V}}$) is a compact subset of $\mathcal{V}$. Hence, by the standard methods of the theory of attractors, $S(t)$ possesses a connected global attractor $\mathcal{A} \subset z_0 + \mathcal{K}_{\mathcal{V}}$.

The proof of Corollary 3.5 is carried out by differentiating the equation with respect to time, and applying the techniques above to the new equation.

6. Proof of Theorem 3.6

Having the growth restriction (3.2), we readily get from Lemma 5.2 the following result.

Theorem 6.1. *The dissipative estimate*

$$\mathcal{E}_{\mathcal{H}}(t) + \int_t^\infty \|\partial_t u(\tau)\|_1^2 d\tau \le \mathcal{Q}(\mathcal{E}_{\mathcal{H}}(0))e^{-\varepsilon t} + C,$$

holds for every $t \ge 0$ and some $\varepsilon > 0$.

Thus, in analogy with the previous case, there exists a bounded absorbing set $\mathbb{B}_{\mathcal{H}}$ for the semigroup $S(t)$ acting on $\mathcal{H}$. In the sequel, we consider initial data $z \in \mathbb{B}_{\mathcal{H}}$. Accordingly, the generic positive constant C that will appear in the forthcoming proofs may depend on $\mathbb{B}_{\mathcal{H}}$. In particular,

$$\mathcal{E}_{\mathcal{H}}(t) + \int_0^\infty \|\partial_t u(\tau)\|_1^2 d\tau \le C. \tag{6.1}$$

For further use, it is convenient to define

$$\varpi(t) = \min\{t, 1\}$$

and

$$\mathcal{F}(t) = \int_0^t \mu(s)\|\partial_t u(t-s)\|_1^2 \mathrm{d}s.$$

Changing the order of integration and using (6.1), we have

$$\int_0^\infty \mathcal{F}(t)\mathrm{d}t = \kappa \int_0^\infty \|\partial_t u(t)\|_1^2 \mathrm{d}t \le C. \tag{6.2}$$

The next two lemmata provide a suitable regularity for the time-derivatives of u.

Lemma 6.2. *For every $t > 0$,*

$$\varpi(t)\|\partial_t u(t)\|_1^2 + \int_0^t \varpi(\tau)\|\partial_{tt} u(\tau)\|^2 \mathrm{d}\tau \le C.$$

Proof. We introduce the functional

$$\Lambda_1 = \|\partial_t u\|_1^2 + 2\langle u, \partial_t u\rangle_1 + 2\langle \varphi(u), \partial_t u\rangle - 2\langle f, \partial_t u\rangle + 2\langle \eta, \partial_t u\rangle_{1,\mu}.$$

Multiplying the first equation of (3.1) by $\partial_{tt} u$, we obtain

$$\frac{\mathrm{d}}{\mathrm{d}t}\Lambda_1 + 2\|\partial_{tt} u\|^2 = 2\langle \partial_t \eta, \partial_t u\rangle_{1,\mu} + 2\langle \varphi'(u)\partial_t u, \partial_t u\rangle + 2\|\partial_t u\|_1^2.$$

The growth restriction (3.2) and the bound (6.1) yield

$$2\langle \varphi'(u)\partial_t u, \partial_t u\rangle \leq C(1 + \|u\|_1^6)\|\partial_t u\|_1^2 \leq C\|\partial_t u\|_1^2.$$

From the second equation of (3.1), we have the equality

$$2\langle \partial_t \eta, \partial_t u\rangle_{1,\mu} = 2\langle T\eta, \partial_t u\rangle_{1,\mu} + 2\kappa\|\partial_t u\|_1^2.$$

Controlling the first term in the rhs by means of Lemma 4.3 with $\varepsilon = 1$, and using again (6.1), we end up with the estimate

$$\frac{\mathrm{d}}{\mathrm{d}t}\Lambda_1 + 2\|\partial_{tt} u\|^2 \leq C(1+\mu)\|\partial_t u\|_1^2 + Q(C\mu + \Gamma_1[\eta] + \mathcal{F}).$$

By virtue of (4.2), the functional

$$\Lambda_2 = \mathcal{E}_{\mathcal{H}} + 2\Phi(u) - 2\langle f, u\rangle$$

fulfills the differential equality

$$\frac{\mathrm{d}}{\mathrm{d}t}\Lambda_2 + 2\|\partial_t u\|_1^2 = -\Gamma_1[\eta]. \tag{6.3}$$

Then, we define

$$\Lambda = \Lambda_1 + Q\Lambda_2,$$

which, due to (3.2) and (6.1), satisfies the bounds

$$\frac{1}{2}\|\partial_t u\|_1^2 - C \leq \Lambda \leq C\|\partial_t u\|_1^2 + C. \tag{6.4}$$

Collecting the above relationships, we learn that

$$\frac{\mathrm{d}}{\mathrm{d}t}\Lambda + 2\|\partial_{tt} u\|^2 \leq C(1+\mu)\|\partial_t u\|_1^2 + C(\mu + \mathcal{F}). \tag{6.5}$$

Assume first $t \in (0,1]$. Multiplying (6.5), written for $t = \tau$, by τ and integrating on $[0,t]$, we find, in light of (6.1) and (6.2),

$$t\Lambda(t) + 2\int_0^t \tau\|\partial_{tt} u(\tau)\|^2 \mathrm{d}\tau \leq C + \int_0^t \Lambda(\tau)\mathrm{d}\tau \leq C.$$

Here, we used the fact that

$$\sup_{\tau>0} \tau\mu(\tau) < \infty,$$

which is an easy consequence of the assumptions on μ. Therefore, (6.4) gives

$$\frac{t}{2}\|\partial_t u(t)\|_1^2 + 2\int_0^t \tau\|\partial_{tt} u(\tau)\|^2 \leq C, \tag{6.6}$$

which proves the case $t \leq 1$. Conversely, if $t > 1$, taking advantage of (6.1) and (6.2), we integrate (6.5) on $[1, t]$, so to get

$$\Lambda(t) + 2\int_1^t \|\partial_{tt}u(\tau)\|^2 \mathrm{d}\tau \leq C + \Lambda(1).$$

By means of (6.4) and (6.6), this implies that

$$\frac{1}{2}\|\partial_t u(t)\|_1^2 + 2\int_1^t \|\partial_{tt}u(\tau)\|^2 \leq C. \tag{6.7}$$

The conclusion for the case $t > 1$ follows by collecting (6.6) and (6.7). □

Lemma 6.3. *For every $t \geq 1$,*

$$\|\partial_{tt}u(t)\| \leq C.$$

Proof. We set

$$m = \sup_{s>0} \big[\varpi(s)(1 + \mu(s))\big] < \infty.$$

Calling $p = \partial_t u$, we introduce the functional

$$\Lambda_1 = \|\partial_t p\|^2 + (1+\kappa)\|p\|_1^2,$$

From Lemma 6.2,

$$\int_0^t \varpi(\tau)\Lambda_1(\tau)\mathrm{d}\tau \leq C. \tag{6.8}$$

A differentiation of the first equation of (3.1) with respect to t gives

$$\partial_{tt}p + A\partial_t p + Ap + \int_0^\infty \mu(s)A\partial_t\eta(s)\mathrm{d}s + \varphi'(u)\partial_t u = 0.$$

Multiplying by $\partial_t p$, and using the second equation of (3.1), we obtain the differential equality

$$\frac{\mathrm{d}}{\mathrm{d}t}\Lambda_1 + 2\|\partial_t p\|_1^2 = -2\langle T\eta, \partial_t p\rangle_{1,\mu} - 2\langle \varphi'(u)\partial_t u, \partial_t p\rangle.$$

On account of (3.2), (6.1) and Lemma 4.3 with $\varepsilon = \varpi(t)/m$, we find the controls

$$-2\langle \varphi'(u)\partial_t u, \partial_t p\rangle \leq \|\partial_t p\|_1^2 + C\|\partial_t u\|_1^2,$$

and

$$-2\langle T\eta, \partial_t p\rangle_{1,\mu} \leq \|\partial_t p\|_1^2 + \frac{C}{\varpi}(\mu + \Gamma_1[\eta] + \mathcal{F}).$$

Therefore,

$$\varpi\frac{\mathrm{d}}{\mathrm{d}t}\Lambda_1 \leq C\|\partial_t u\|_1^2 + C(\mu + \Gamma_1[\eta] + \mathcal{F}).$$

Taking now Λ_2 as in the proof of Lemma 6.2, and recalling (6.3), we are led to

$$\varpi\frac{\mathrm{d}}{\mathrm{d}t}\Lambda_1 + \frac{\mathrm{d}}{\mathrm{d}t}C\Lambda_2 \leq C(\mu + \mathcal{F}). \tag{6.9}$$

Writing (6.9) for $t = \tau$, multiplying by τ and integrating on $[0, 1]$, we obtain the inequality

$$\Lambda_1(1) \leq C + 2\int_0^1 \tau\Lambda_1(\tau)\mathrm{d}\tau \leq C,$$

thanks to (6.1), which in turn implies that $|\Lambda_2| \leq C$, (6.2) and (6.8). Finally, if $t \geq 1$, integrating (6.9) on $[1, t]$ and using (6.1) and (6.2), we arrive at

$$\Lambda_1(t) \leq C + \Lambda_1(1) \leq C.$$

Hence, we conclude that

$$\|\partial_{tt} u(t)\| = \|\partial_t p(t)\| \leq C,$$

for every $t \geq 1$. ☐

At this point, we define

$$\varphi_\omega(x) = \varphi(x) + \omega x,$$

with ω given by (2.4), so that $\varphi_\omega'(x) \geq 0$, and

$$h(t) = f - \partial_{tt} u(t).$$

In light of Lemma 6.3,

$$\sup_{t \geq 1} \|h(t)\| \leq C. \tag{6.10}$$

Accordingly, the first equation of (3.1) may be rewritten as

$$A\partial_t u + Au + \varphi_\omega(u) + \int_0^\infty \mu(s) A\eta(s)\mathrm{d}s = h + \omega u.$$

Next, we consider the splitting

$$(u, \eta) = (v, \xi) + (w, \zeta),$$

where (v, ξ) and (w, ζ) are the solutions of the Cauchy problems on $[1, \infty)$

$$\begin{cases} A\partial_t v + Av + \varphi_\omega(u) - \varphi_\omega(w) + \displaystyle\int_0^\infty \mu(s) A\xi(s)\mathrm{d}s = 0, \\ \partial_t \xi = T\xi + \partial_t v, \\ (v(1), \xi^1) = (u(1), \eta^1), \end{cases} \tag{6.11}$$

and

$$\begin{cases} A\partial_t w + Aw + \varphi_\omega(w) + \displaystyle\int_0^\infty \mu(s) A\zeta(s)\mathrm{d}s = h + \omega u, \\ \partial_t \zeta = T\zeta + \partial_t w, \\ (w(1), \zeta^1) = 0. \end{cases} \tag{6.12}$$

Lemma 6.4. *For every $t \geq 1$,*

$$\|v(t)\|_1^2 + \|\xi^t\|_{1,\mu}^2 \leq Ce^{-\varepsilon t},$$

for some $\varepsilon > 0$.

Proof. Setting

$$\Lambda = \|v\|_1^2 + \Psi_1[\xi],$$

using (4.3) and multiplying the first equation of (6.11) by v, from the monotonicity of φ_ω we readily get the inequality

$$\frac{\mathrm{d}}{\mathrm{d}t}\Lambda + 2\|v\|_1^2 + \|\xi\|_{1,\mu}^2 \leq 0.$$

which, by means of (4.4), turns into

$$\frac{\mathrm{d}}{\mathrm{d}t}\Lambda + \varepsilon\Lambda \leq 0,$$

for some $\varepsilon > 0$. On account of (6.1), the Gronwall lemma and a further application of (4.4) entail

$$\|v(t)\|_1^2 \leq \Lambda(t) \leq \Lambda(1)e^{-\varepsilon(t-1)} \leq C(\|u(1)\|_1^2 + \|\eta^1\|_{1,\mu}^2)e^{-\varepsilon t} \leq Ce^{-\varepsilon t}$$

The conclusion follows by applying Lemma 4.1 to $\psi^t = \xi^{t+1}$. □

Lemma 6.5. *For every $t \geq 1$,*

$$\|w(t)\|_2 + \|\zeta^t\|_{2,\mu} \leq C.$$

Proof. Setting

$$\Lambda = \|w\|_2^2 + \Psi_2[\zeta],$$

using (4.3) and multiplying the first equation of (6.12) by Aw, appealing again to the monotonicity of φ_ω, we are led to

$$\frac{\mathrm{d}}{\mathrm{d}t}\Lambda + 2\|w\|_2^2 + \|\zeta\|_{2,\mu}^2 \leq 2\langle h, Aw\rangle + 2\omega\langle u, Aw\rangle \leq \|w\|_2^2 + C,$$

where the latter inequality follows from (6.1) and (6.10). Hence, arguing exactly as in the former proof, we obtain the desired result. □

Lemma 6.6. *The inequality*

$$\|\zeta^t\|_{2,\mu;1} + \sup_{s>0}\|\zeta^t(s)\|_2 \leq C.$$

holds for every $t \geq 1$.

Proof. Apply Lemma 4.2 to $\psi^t = \zeta^{t+1}$, noting that the estimates of Lemma 6.5 furnish the further control

$$\|\partial_t w(t)\|_2 \leq C, \qquad \forall t \geq 1,$$

by comparison in the equation. □

Summarizing the above results, for every $z \in \mathbb{B}_{\mathcal{H}}$, the solution

$$S(t)z = (v(t) + w(t), \partial_t u(t), \xi^t + \zeta^t)$$

fulfills, for every $t \geq 1$, the relations

$$\|v(t)\|_1^2 + \|\xi^t\|_{1,\mu}^2 \leq Ce^{-\varepsilon t}$$

and

$$\|w(t)\|_2 + \|\partial_t u(t)\|_1 + \|\zeta^t\|_{2,\mu;1} + \sup_{s>0}\|\zeta^t(s)\|_2 \leq C.$$

This means that $S(t)\mathbb{B}_{\mathcal{H}}$ is (exponentially) attracted by the set

$$\mathcal{K}_{\mathcal{H}} = \big\{(\bar{u}, \bar{v}, \bar{\eta}) : \|\bar{u}\|_2 + \|\bar{v}\|_1 + \|\bar{\eta}\|_{2,\mu;1} + \sup_{s>0}\|\bar{\eta}(s)\|_2 \leq C,\ \bar{\eta}(0) = 0\big\}.$$

which is compact in $\mathcal{H}$ by [24, Lemma 5.5]. Therefore, the semigroup $S(t)$ on $\mathcal{H}$ possesses a connected global attractor $\mathcal{A}_{\mathcal{H}} \subset \mathcal{K}_{\mathcal{H}}$. On the other hand, $\mathcal{K}_{\mathcal{H}}$, and so $\mathcal{A}_{\mathcal{H}}$, is bounded in $\mathcal{V}$. Thus, from Theorem 3.3, $S(t)\mathcal{A}_{\mathcal{H}}$ is attracted by $\mathcal{A}$ in the norm of $\mathcal{V}$. But $\mathcal{A}_{\mathcal{H}} = S(t)\mathcal{A}_{\mathcal{H}}$, which forces the equality $\mathcal{A}_{\mathcal{H}} = \mathcal{A}$. The proof of Theorem 3.6 is finished.

References

[1] A.V. Babin, M.I. Vishik, *Attractors of evolution equations*, North-Holland, Amsterdam, 1992.

[2] S. Borini, V. Pata, *Uniform attractors for a strongly damped wave equation with linear memory.* Asymptot. Anal. **20** (1999), 263–277.

[3] A.N. Carvalho, J.W. Cholewa, *Local well posedness for strongly damped wave equations with critical nonlinearities*, Bull. Austral. Math. Soc. **66** (2002), 443–463.

[4] A.N. Carvalho, J.W. Cholewa, *Attractors for strongly damped wave equations with critical nonlinearities*, Pacific J. Math. **207** (2002), 287–310.

[5] V.V. Chepyzhov, V. Pata, *Some remarks on stability of semigroups arising from linear viscoelasticity,* Asymptot. Anal. **46** (2006), 251–273.

[6] C.M. Dafermos, *Asymptotic stability in viscoelasticity*, Arch. Rational Mech. Anal. **37** (1970), 554–569.

[7] A. Eden, V. Kalantarov, *Finite dimensional attractors for a class of semilinear wave equations*, Turkish J. Math. **20** (1996), 425–450.

[8] G.A. Francfort, P.M. Suquet, *Homogenization and mechanical dissipation in thermoviscoelasticity,* Arch. Rational Mech. Anal. **96** (1986), 265–293.

[9] S. Gatti, A. Miranville, V. Pata, S. Zelik, *Attractors for semilinear equations of viscoelasticity with very low dissipation*, Rocky Mountain J. Math. (to appear).

[10] J.M. Ghidaglia, A. Marzocchi, *Longtime behaviour of strongly damped wave equations, global attractors and their dimension*, SIAM J. Math. Anal. **22** (1991), 879–895.

[11] C. Giorgi, A. Marzocchi, V. Pata, *Asymptotic behavior of a semilinear problem in heat conduction with memory,* NoDEA Nonlinear Differential Equations Appl. **5** (1998), 333–354.

[12] M. Grasselli, V. Pata, *Longtime behavior of a homogenized model in visco-elastodynamics*, Discrete Contin. Dynam. Systems **4** (1998), 339–358.

[13] M. Grasselli, V. Pata, *Uniform attractors of nonautonomous systems with memory,* in "Evolution Equations, Semigroups and Functional Analysis" (A. Lorenzi and B. Ruf, Eds.), pp.155–178, Progr. Nonlinear Differential Equations Appl. no.50, Birkhäuser, Boston, 2002.

[14] J.K. Hale, *Asymptotic behavior of dissipative systems*, Amer. Math. Soc., Providence, 1988.

[15] A. Haraux, *Systèmes dynamiques dissipatifs et applications*, Masson, Paris, 1991.

[16] M. Houshiyari, I. Erdem, *Global attractors for nonlinear pseudoparabolic and pseudohyperbolic equations*, Hacet. Bull. Nat. Sci. Eng. Ser. B **27** (1998), 19–31.

[17] V.K. Kalantarov, *Global behavior of solutions to nonlinear problems of mathematical physics of classical and nonclassical type*, Postdoc Thesis, Leningrad, 1988.

[18] V.K. Kalantarov, *Attractors for some nonlinear problems of mathematical physics*, J. Soviet Math. **40** (1988), 619–622.

[19] O.A. Ladyzhenskaya, *Attractors for semigroups and evolution equations*, Cambridge University Press, Cambridge, UK, 1991.

[20] M.-L. Mascarenhas, *Homogenisation of a viscoelastic equation with non-periodic coefficients*, Proc. Roy. Soc. Edinburgh Sect. A **106** (1987), 143–160.

[21] P. Massat, *Limiting behavior for strongly damped nonlinear wave equations*, J. Differential Equations **48** (1983), 334–349.

[22] V. Pata, M. Squassina, *On the strongly damped wave equation*, Comm. Math. Phys. **253** (2005), 511–533.

[23] V. Pata, S. Zelik, *Smooth attractors for strongly damped wave equations*, Nonlinearity **19** (2006), 1495–1506.

[24] V. Pata, A. Zucchi, *Attractors for a damped hyperbolic equation with linear memory,* Adv. Math. Sci. Appl. **11** (2001), 505–529.

[25] R. Temam, *Infinite-dimensional dynamical systems in mechanics and physics*, Springer, New York, 1997.

[26] G.F. Webb, *Existence and asymptotic behavior for a strongly damped nonlinear wave equation,* Canad. J. Math. **32** (1980), 631–643.

[27] S. Zhou, *Global attractor for strongly damped nonlinear wave equations*, Funct. Differ. Equ. **6** (1999), 451–470.

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