UPPER AND LOWER BOUNDS FOR THE KOLMOGOROV ENTROPY OF THE ATTRACTOR FOR AN RDE IN AN UNBOUNDED DOMAIN.

M.A. Efendiev and S.V. Zelik

Abstract. The long-time behaviour of bounded solutions of a reaction-diffusion system in an unbounded domain $\Omega \subset \mathbb{R}^n$, for which the nonlinearity $f(u, \nabla_x u)$ explicitly depends on $\nabla_x u$ is studied. We prove the existence of a global attractor whose fractal dimension is infinite, and give upper and lower bounds for the Kolmogorov entropy of the attractor and analyze the sharpness of these bounds.

Introduction

In this paper, quasilinear second-order parabolic equations and systems of reaction-diffusion type

$$
\begin{cases} \partial_t u - \Delta_x u + f(u, \nabla_x u) + \lambda_0 u = g; \; x \in \Omega \\ u\big|_{t=0} = u_0 \,, \quad u\big|_{\partial\Omega} = 0 \end{cases} \tag{0.1}
$$

are considered.

Here $\Omega \subset \mathbb{R}^n$ is an unbounded domain in $\mathbb{R}^n$ with a sufficiently smooth boundary (see §1), $u = (u^1, \cdots, u^k)$ is an unknown vector-valued function, Δ_x is the Laplacian with respect to $x = (x_1, \cdots, x_n)$, f and g are given functions and λ_0 is a fixed positive constant.

It is assumed also that the nonlinear term $f(u, \nabla_x u)$ satisfies the conditions

$$
\begin{cases} 1.\, f \in C^1(\mathbb{R}^k \times \mathbb{R}^{nk}, \mathbb{R}^k) \\ 2.\, f(u, \nabla_x u).u \geq -C \\ 3.\, |f(u, \nabla_x u)| + |f'_u(u, \nabla_x u)| \leq Q(|u|)(1 + |\nabla_x u|^r) \;\; 0 \leq r < 2 \\ 4.\, |f'_{\nabla_x u}(u, \nabla_x u)| \leq Q(|u|)(1 + |\nabla_x u|^{r-1}) \end{cases} \tag{0.2}
$$

Here and below we denote by $u \cdot v$ the inner product in the space R^k.

It is well known that in many cases the longtime behavior of a dynamical system, generated by evolutionary equations of mathematical physics, can be naturally described in terms of attractors of the corresponding semigroup (see [2], [10], [18]). In bounded domains, the existence of the attractor has been established for a large class of equations such as reaction-diffusion equations, nonlinear wave equations, the 2D Navier–Stokes system, and many others. Under some natural assumptions, for all the equations mentioned above, it has been proved that the attractor in the autonomous case has finite Hausdorff and fractal dimension (see [10], [18]).

The equations of mathematical physics which depend explicitly on t in bounded domains Ω are considered in [4], [5]. Moreover, it has been shown there that if the

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time-dependence of the right-hand side is in a certain sense "infinite dimensional" (for example, when the right–hand side is almost-periodic in t with an infinite number of independent frequences), then the uniform attractor of the corresponding equation naturally has infinite Hausdorff and fractal dimension.

Thus – in contrast to the autonomous case – in the nonautonomous one the fractal dimension is not a convenient quantitative characteristic of the "size" of attractors and consequently the question of finding another measure of the "size" arises.

One of possible approaches to handle this problem, which has been suggested in [5], is to estimate Kolmogorov's ε-entropy of the attractor. Recall, that by definition Kolmogorov's ε-entropy $\mathbb{H}_\varepsilon(\mathcal{A})$ of an attractor $\mathcal{A}$ is the logarithm of the minimal number $N_\varepsilon(\mathcal{A})$ of ε-balls in the appropriate phase space which cover the attractor:

$$\mathbb{H}_\varepsilon(\mathcal{A}) = \ln N_\varepsilon(\mathcal{A}) \tag{0.3}$$

Note that since $\mathcal{A}$ is compact then (0.3) is well defined and finite for every $\varepsilon > 0$.

For unbounded domains Ω, the behavior of solutions for (0.1) becomes much more complicated. In this case even the problem of finding the appropriate phase space for (0.1) becomes nontrivial. For instance, in [1], [3], [8] this equation has been studied in weighted Sobolev spaces $W^{l,p}_\phi(\Omega)$ with $\phi(x) = \phi_\alpha(x) = (1 + |x|^2)^{\alpha/2}$. The case of general weights ϕ is considered in [9].

In the present paper, we assume that the solution $u(t, x)$ is bounded as $|x| \to \infty$. To be more precise it is assumed that for every fixed $t \geq 0$

$$u(t) \in W^{l,p}_b(\Omega) \equiv \{v : \|v\|_{W^{l,p}_b} = \sup_{x_0 \in \Omega} \|v\|_{W^{l,p}(\Omega \cap B^1_{x_0})} < \infty\} \tag{0.4}$$

with the appropriate exponents l and p. (Here and below we denote by $B^R_{x_0}$ the R-ball in $\mathbb{R}^k$ centered in x_0.)

In the autonomous case $g = g(x)$ reaction-diffusion equations and systems of the type (0.1) under the assumptions (0.4) are considered in [6], [7], [15], [16], [17], [23], [24].

Recall that under the above assumptions the attractor $\mathcal{A}$ of the equation (0.1) may have (and has in general) infinite Hausdorff and fractal dimension even in the autonomous case (see [6], [9], [23], [24]). Thus, in contrast to the case of bounded domains where the infinite dimensional attractor can appear only in the nonautonomous case and only due to the "infinite dimensional" external time-dependent forces, in the case where Ω is unbounded, the infinite dimensionality appears even in the autonomous case and has consequently the internal nature.

Note that in general the attractor $\mathcal{A}$ of the problem (0.1) is not compact in the uniform topology of the space (0.4) but only in a local topology of the space $W^{l,p}_{loc}(\Omega)$. That is why the Kolmogorov entropy of an attractor $\mathcal{A}$ in a *weighted* Sobolev space $W^{l,p}_{e^{-|x|}}$ with an exponential decaying weight was considered in [23] (since the attractor is bounded in the space $W^{2,p}_b(\Omega)$ and compact in $W^{2,p}_{loc}(\Omega)$ then it is easy to verify that it is compact in $W^{2,p}_{e^{-|x|}}(\Omega)$ as well and consequently it's ε-entropy is well defined).

It is proved there that if $\Omega = \mathbb{R}^n$ and the nonlinearity f does not depends explicitly on a gradient and (0.2) is valid then this entropy possesses the estimate

$$\mathbb{H}_\varepsilon(\mathcal{A}, W^{l,p}_{e^{-|x|}}(\mathbb{R}^n)) \leq C \left(\ln \frac{1}{\varepsilon}\right)^{n+1} \tag{0.5}$$

Moreover, the examples which admit a lower bound of the entropy with the same type of asymptotics were also constructed.

The entropy per unit volume for the attractors of complex Ginzburg-Landau equations in $\mathbb{R}^n$, $n \le 3$ has been considered in [6]. It is proved there that

$$C_1 \ln \frac{1}{\varepsilon} \le \overline{\mathbb{H}}_\varepsilon(\mathcal{A}) \equiv \lim_{R\to\infty} \frac{\mathbb{H}_\varepsilon(\mathcal{A}\big|_{B_0^R})}{R^n} \le C_2 \ln \frac{1}{\varepsilon} \tag{0.6}$$

The topological entropy per unit volume for RDE in $\mathbb{R}^n$ has been introduced in [7].

A systematic study of the entropy $H_\varepsilon(\mathcal{A}\big|_{\Omega\cap B_{x_0=}^R})$ and it's dependence on three parameters R,ε, and x_0 of the attractors of the autonomous and nonautonomous reaction-diffusion equations in unbounded domains $\Omega \subset \mathbb{R}^n$ was given in [24] for the case $f = f(u)$. It was shown particularly that in the autonomous case $g = g(x)$ the entropy of the attractor $\mathcal{A}$ possesses the following estimate:

$$\mathbb{H}_\varepsilon(\mathcal{A}\big|_{\Omega\cap B_{x_0}^R}) \le C\,\mathrm{vol}_{x_0,\Omega}(R + K \ln \frac{1}{\varepsilon}) \ln \frac{1}{\varepsilon} \tag{0.7}$$

where $\mathrm{vol}_{x_0,\Omega}(r) = \mathrm{vol}(\Omega \cap B_{x_0}^r)$, $\mathrm{vol}(\cdot)$= means the n-dimensional volume, and constants C, K are independent of R, ε and x_0. Moreover, if $\Omega = \mathbb{R}^n$, then (0.5) implies that

$$\mathbb{H}_\varepsilon(\mathcal{A}\big|_{B_{x_0}^R}) \le C(R + K \ln \frac{1}{\varepsilon})^n \ln \frac{1}{\varepsilon} \tag{0.8}$$

The lower bounds of these values were also obtained for the case where $\Omega = \mathbb{R}^n$. It is proved there that for a rather wide class of the nonlinearities $f = f(u)$ which include for example the Chafee-Infante equation, Ginzburg-landau equations, and related equations, this entropy possesses the following lower estimates.

$$\mathbb{H}_\varepsilon(\mathcal{A}\big|_{B_{x_0}^R}) \ge C_1 R^n \ln \frac{1}{\varepsilon} \tag{0.9}$$

for $R \ge R_0$ and $\varepsilon < \varepsilon_0$, and consequently the estimate (0.8) is sharp if $R \sim \ln \frac{1}{\varepsilon}$ or $R \gg \ln \frac{1}{\varepsilon}$. For the case where $R \ll \ln \frac{1}{\varepsilon}$ (particularly for $R = 1$) it is proved that for every $\delta > 0$ there exists $C_\delta > 0$ such that

$$\mathbb{H}_\varepsilon(\mathcal{A}\big|_{B_{x_0}^1}) \ge C_\delta \left(\ln \frac{1}{\varepsilon}\right)^{n+1-\delta} \tag{0.10}$$

The main aim of this paper is to extend all these results to the case when the nonlinear term may depend explicitly on a gradient $\nabla_x u$. It will be shown below that the estimates (0.7)–(0.10) remain valid for reaction-diffusion equations which depend explicitly on a gradient $\nabla_x u$. Moreover, it has been recently proved that these estimates hold also for a wide class of damped nonlinear wave equations in unbounded domains (see [25]). Thus, the estimates (0.7) – (0.10) have a universal nature. From the other side, if Ω is bounded then (0.7) implies that

$$\mathbb{H}_\varepsilon(\mathcal{A}, W^{2,p}(\Omega)) \le C\,\mathrm{vol}(\Omega) \ln \frac{1}{\varepsilon}$$

which reflects the well-known heuristic principle that the equations of mathematical physics in bounded domains have finite-dimensional attractors. Thus, these estimates may be considered as a natural generalization of this principle to the case of unbounded domains. Note also that the estimates (0.7)–(0.10) imply particularly as the estimates (0.5) as the estimates (0.6) obtained earlier.

The present paper is organized as follows.

Some auxiliary result about the weighted Sobolev spaces, spaces (0.4) and the regularity in an unbounded domains which will be used throughout the paper is formulated in Section 1.

The existence of a solution for the equation (0.1) in the appropriate phase space (0.4) is proved in Section 2.

The uniqueness of this solution and a number of the estimates for a difference between two solutions of our problem are obtained in Section 3. These estimate are of fundamental significance in our further study the entropy of the attractor.

The attractor for the nonlinear reaction diffusion equation (0.1) is constructed in Section 4 in the phase space (0.4).

The definition and a number of useful examples of the entropy for typical sets in function spaces are given in Section 5.

The upper and lower bounds for the Kolmogorov entropy of an attractor, as constructed above are obtained in Sections 6 and 7 correspondingly.

§1 Functional spaces and regularity theorems.

In this Section we introduce several clasess of Sobolev spaces in unbounded domains and recall shortly some of their properties which will be essentially used below. For a detailed study of these spaces see [9], [24].

Definition 1.1. *A function $\phi \in L^\infty_{loc}(\mathbb{R}^n)$ is called a weight function with the rate of growth $\mu \geq 0$ if the condition*

$$\phi(x+y) \leq C_\phi e^{\mu|x|}\phi(y), \ \ \phi(x) > 0 \tag{1.1}$$

is satisfied for every $x, y \in \mathbb{R}^n$.

Remark 1.1. *It is not difficult to deduce from* (1.1) *that*

$$\phi(x+y) \geq C_\phi^{-1} e^{-\mu|x|}\phi(y) \tag{1.2}$$

is also satisfied for every $x, y \in \mathbb{R}^n$.

Proposition 1.1. *Let ϕ_1 and ϕ_2 be weight functions with the rates of growth μ_1 and μ_2 correspondingly. Then,*

1. $\alpha\phi_1 + \beta\phi_2$, $\max\{\phi_1, \phi_2\}$, and $\min\{\phi_1, \phi_2\}$ are the weight functions with the rate of growth $\max\{\mu_1, \mu_2\}$ for every $\alpha, \beta > 0$.

2. $\phi_1 \cdot \phi_2$ and $\phi_1 \cdot (\phi_2)^{-1}$ are the weight functions with the rate of growth $\mu_1 + \mu_2$.

3. $(\phi_1)^\alpha$ is the weight function with the rate of growth $|\alpha|\mu_1$.

The assertions of this proposition are immediate corollaries of (1.1) and (1.2).

The following example of weight functions are of fundamental significance for our purposes:

$$\phi_{\{\varepsilon\},x_0}(x) = e^{-\varepsilon|x-x_0|}, \ \ \varepsilon \in \mathbb{R}, \ \ x_0 \in \mathbb{R}^n$$

(Evidently this weight has the rate of growth $|\varepsilon|$.)

Definition 1.2. *Let $\Omega \subset \mathbb{R}^n$ be some (unbounded) domain in $\mathbb{R}^n$ and let ϕ be a weight function with the rate of growth μ. Define the space*

$$L^p_\phi(\Omega) = \left\{ u \in D'(\Omega) \ : \ \|u, \Omega, \|_{\phi,0,p} \equiv \int_\Omega \phi(x)|u(x)|^p\,dx < \infty \right\}$$

Analogously the weighted Sobolev space $H^{l,p}_\phi(\Omega)$, $l \in \mathbb{N}$ is defined as the space of distributions whose derivatives up to the order l inclusively belong to $L^p_\phi(\Omega)$.

For the simplicity of notations we will right throughout of the paper $W^{s,p}_{\{\varepsilon\}}$ instead of $W^{s,p}_{e^{-\varepsilon|x|}}$.

We define also another class of weighted Sobolev spaces

$$W^{l,p}_{b,\phi}(\Omega) = \left\{ u \in D'(\Omega) \ : \|u, \Omega\|^p_{b,\phi,l,p} = \sup_{x_0 \in \mathbb{R}^n} \phi(x_0)\|u, \Omega \cap B^1_{x_0}\|^p_{l,p} < \infty \right\}$$

Here and below we denote by $B^R_{x_0}$ the ball in $\mathbb{R}^n$ of radius R, centered in x_0, and $\|u, V\|_{l,p}$ means $\|u\|_{W^{l,p}(V)}$.

We will write $W^{l,p}_b$ instead of $W^{l,p}_{b,1}$.

Proposition 1.1.

1. Let $u \in L^p_\phi(\Omega)$, where ϕ is a weight function with the rate of growth μ. Then for any $1 \le q \le \infty$ the following estimate is valid

$$\left(\int_\Omega \phi(x_0)^q \left(\int_\Omega e^{-\varepsilon|x-x_0|}|u(x)|^p\,dx \right)^q dx_0 \right)^{1/q} \le C \int_\Omega \phi(x)|u(x)|^p\,dx \tag{1.3}$$

for every $\varepsilon > \mu$, where the constant C depends only on ε, μ and C_ϕ from (1.1) *(and independent of Ω).*

2. Let $u \in L^\infty_\phi(\Omega)$. Then the following analogue of the estimate (1.3) *is valid*

$$\sup_{x_0 \in \Omega} \left\{ \phi(x_0) \sup_{x \in \Omega}\{e^{-\varepsilon|x-x_0|}|u(x)|\} \right\} \le C \sup_{x \in \Omega}\{\phi(x)|u(x)|\} \tag{1.4}$$

The proof of this Proposition can be found in [9] or [24].

For the more detailed study of functional spaces defined above we need some regularity assumptions on the domain $\Omega \subset \mathbb{R}^n$ which are assumed to be valid throughout of the paper.

We suppose that there exists a positive number $R_0 > 0$ such that for every point $x_0 \in \Omega$ there exists a smooth domain $V_{x_0} \subset \Omega$ such that

$$B^{R_0}_{x_0} \cap \Omega \subset V_{x_0} \subset B^{R_0+1}_{x_0} \cap \Omega \tag{1.5}$$

Moreover it is assumed also that there exists a diffeomorphizm $\theta_{x_0} : B^1_0 \to V_{x_0}$ such that $\theta_{x_0}(x) = x_0 + p_{x_0} = (x)$ and

$$\|p_{x_0}\|_{C^N} + \|p^{-1}_{x_0}\|_{C^N} \le K \tag{1.6}$$

where the constant K is assumed to be independent of $x_0 \in \Omega$ and N is large enough. For simplicity we suppose below that (1.5) and (1.6) hold for $R_0 = 2$.

Note that in the case when Ω is bounded the conditions (1.5) and (1.6) are equivalent to the condition: the boundary $\partial\Omega$ is a smooth manifold, but for unbounded domains the only smoothness of the boundary is not sufficient to obtain the regular structure of Ω when $|x| \to \infty$ since some uniform with respect to $x_0 \in \Omega$ smoothness conditions are required. It is the most convenient for us to formulate these conditions in the form (1.5) and (1.6).

Proposition 1.2. *Let the domain Ω satisfy the conditions* (1.5) *and* (1.6)*, the weight function – the condition* (1.1) *and let R be some positive number. Then the following estimates are valid*

$$C_2 \int_{\Omega} \phi(x)|u(x)|^p \, dx \le \int_{\Omega} \phi(x_0) \int_{\Omega \cap B^R_{x_0}} |u(x)|^p \, dx \, dx_0 \le C_1 \int_{\Omega} \phi(x)|u(x)|^p \, dx \tag{1.7}$$

Proof. Let us change the order of integration in the middle part of (1.7)

$$\int_{\Omega} \phi(x_0) \int_{\Omega \cap B^R_{x_0}} |u(x)|^p \, dx \, dx_0 = \int_{\Omega} |u(x)|^p \left(\int_{\Omega} \chi_{\Omega \cap B^R_x}(x_0)\phi(x_0) \, dx_0 \right) dx \tag{1.8}$$

Here $\chi_{\Omega \cap B^R_x}$ is the characteristic function of the set $\Omega \cap B^R_x$.

It follows from the inequalities (1.1) and (1.2) that

$$C_1\phi(x) \le \inf_{x_0 \in B^R_x} \phi(x_0) \le \sup_{x_0 \in B^R_x} \phi(x_0) \le C_2\phi(x) \tag{1.9}$$

and the assumptions (1.5) and (1.6) imply that

$$0 < C_1 \le \mathrm{mes}(\Omega \cap B^R_x) \le C_2 \tag{1.10}$$

uniformly with respect to $x \in \Omega$.

The estimate (1.7) is an immediate corollary of the estimates (1.8)–(1.10). Proposition 1.2 is proved. $\Box$

Corollary 1.1. *Let* (1.5) *and* (1.6) *be valid. Then the equivalent norm in weighted Sobolev space $W^{l,p}_{\phi}(\Omega)$ is given by the following expression:*

$$\|u, \Omega\|_{\phi,l,p} = \left(\int_{\Omega} \phi(x_0) \|u, \Omega \cap B^R_{x_0}\|^p_{l,p} \, dx_0 \right)^{1/p} \tag{1.11}$$

Particularly, the norms (1.11) *are equivalent for different $R \in \mathbb{R}_+$.*

To study the equation (0.1) we need also weighted Sobolev spaces with fractional derivatives $s \in \mathbb{R}_+$ (not only $s \in \mathbb{Z}$). For the first we recall (see [19] for details) that if V is a bounded domain the norm in the space $W^{s,p}(V)$, $s = [s] + l$, $0 < l < 1$, $[s] \in \mathbb{Z}_+$ can be given by the following expression

$$\|u, V\|^p_{s,p} = \|u, V\|^p_{[s],p} + \sum_{|\alpha|=[s]} \int_{x \in V} \int_{y \in V} \frac{|D^\alpha u(x) - D^\alpha u(y)|^p}{|x-y|^{n+lp}} \, dx \, dy \tag{1.12}$$

It is not difficult to prove arguing as in Proposition 1.2 and using this representation that for any bounded domain V with a sufficiently smooth boundary

$$\|u, V\|^p_{s,p} \le C_1 \int_{x_0 \in V} \|u, V \cap B^R_{x_0}\|^p_{s,p} \, dx_0 \le C_2 \|u, V\|^p_{s,p} \tag{1.13}$$

This justifies the following definition.

Definition 1.3. *Define the space* $W^{s,p}_\phi(\Omega)$ *for any* $s\in\mathbb{R}_+$ *by the norm* (1.11).

It is not difficult to check that these norms are also equivalent for different $R>0$. Note now that the weight functions

$$\phi_{\{\varepsilon\},x_0}=e^{-\varepsilon|x-x_0|} \tag{1.14}$$

satisfy the conditions (1.1) *uniformly* with respect to $x_0\in\mathbb{R}^n$, consequently all estimates obtained above for the arbitrary weights will be valid for the family (1.14) with constants, independent of $x_0\in\mathbb{R}^n$. Since these estimates are of fundamental significance for us we write it explicitly in a number of corollaries formulated below.=

Corollary 1.2. *Let* $u\in L^p_{\{\delta\}}(\Omega)$ *for* $0<\delta<\varepsilon$. *Then the following estimate holds uniformly with respect to* $y\in\mathbb{R}^n$

$$\left(\int_\Omega e^{-q\delta|x_0-y|}\left(\int_\Omega e^{-\varepsilon|x-x_0|}|u(x)|^p\,dx\right)^q dx_0\right)^{1/q}\le \le C_{\varepsilon,q}\int_\Omega e^{-\delta|x-y|}|u(x)|^p\,dx \tag{1.15}$$

Moreover if $u\in L^\infty_{\{\delta\}}(\Omega)$, $\delta<\varepsilon$ *then*

$$\sup_{x_0\in\Omega}\left\{e^{-\delta|x_0-y|}\sup_{x\in\Omega}\{e^{-\varepsilon|x-x_0|}|u(x)|\}\right\}\le C_{\varepsilon,\delta}\sup_{x\in\Omega}\{e^{-\delta|x-y|}|u(x)|\} \tag{1.16}$$

Corollary 1.3. *Let* $u\in W^{l,p}_{b,\phi}(\Omega)$ *and* ϕ *be a weight function with the rate of growth* $\mu<\varepsilon$. *Then*

$$C_1\|u,\Omega\|^p_{b,\phi,l,p}\le \le\sup_{x_0\in\Omega}\left\{\phi(x_0)\int_{x\in\Omega}e^{-\varepsilon|x-x_0|}\|u,\Omega\cap B^1_x\|^p_{l,p}\,dx\right\}\le C_2\|u,\Omega\|^p_{b,\phi,l,p} \tag{1.17}$$

For the proof of this corollary see [24].

In conclusion of this Section we consider the auxiliary linear problem of type (0.1) and formulate some regularity results which will be useful in studing the nonlinear problem.

Proposition 1.3. *Let* $v(t)\in C([0,T],W^{1,2}_{\{\varepsilon\}}(\Omega))$ *be a solution of the following problem*

$$\begin{cases}\partial_t v-\Delta_x v+\lambda_0 v=g(t);\\ v\big|_{t=0}=v_0;\quad v\big|_{\partial\Omega}=0\end{cases} \tag{1.18}$$

Suppose also that $g\in L^\infty([0,T],L^p_b(\Omega))$ *for some* $1<p<\infty$ *and* $v_0\in W^{2-\delta,p}_b(\Omega)$ *for some* $1>\delta>0$. *Then*

$$v\in L^\infty([0,T],W^{2-\delta,p}_b(\Omega))\cap C^{1-\frac{\delta}{2}}([0,T],L^2_b(\Omega)) \tag{1.19}$$

and the following estimate holds when $t\le T$

$$\|v(t),\Omega\|_{b,2-\delta,p}\le C_1\|v(0),\Omega\|_{b,2-\delta,p}e^{-\mu t}+C_2\sup\nolimits_{s\in[0,t]}\{e^{\mu(s-t)}\|g(s)\|_{b,0,p}\} \tag{1.20}$$

for some $\mu=\mu(\lambda_0)>0$.

The regularity theorem in the spaces $W^{l,p}_b$ for solutions of the linear problem (1.18) can be deduced from the apprpriate regularity theorem in weighted Sobolev spaces in a standard way (see, for instance, [9], [24]).

§2 The nonlinear equation: a priori estimates, existence of solutions.

In this Section we consider the nonlinear parabolic boundary problem (0.1) in the unbounded domain Ω which is assumed to to satisfy the conditions (1.5) and (1.6) formulated in Section 1. We suppose in this Section that the right–hand side $g = g(x)$ is from the space $L^p_b(\Omega)$ for some $p > \max\{2, \frac{n}{2}\}$ and the initial date u_0 – from the space $W_b^{2-\delta,p}(\Omega) \cap \{u_0\big|_{\partial\Omega} = 0\}$, where the exponent $\delta > 0$ will be defined below

A solution of the equation (3.1) is defined to be a function u which belongs to the space

$$\cap_{\varepsilon>0} \left\{ L^p([0,T], W^{2,p}_{\{\varepsilon\}}(\Omega)) \cap W^{1,p}([0,T], L^p_{\{\varepsilon\}}(\Omega)) \right\} \tag{2.1}$$

and satisfies the equation (0.1) in the sense of distributions.

The main aim of this Section is to prove a number of a priori estimates for the solutions of (3.1) and to derive the existence of solutions for this equation.

Theorem 2.1. *Let u be a solution of* (0.1) *and let $\delta < 2 - \frac{n}{p}$. Then the following estimate is valid:*

$$\|u(T)\|_{0,\infty} \le Ce^{-\gamma T}\|u(0)\|_{b,2-\delta,p} + C(1 + \|g\|_{b,0,2}) \tag{2.2}$$

for some positive $\gamma > 0$.

Proof. Let us consider the function $w(t,x) = u(t,x).u(t,x)$. Then due to the equation (0.1)

$$\partial_t w - \Delta_x w + 2\lambda_0 w = -2\nabla_x u.\nabla_x u - 2f(u, \nabla_x u).u + 2g.u \le C + 2g.u \equiv h_u(t) \tag{2.3}$$

We consider also the auxiliary linear problem

$$\begin{cases} \partial_t v - \Delta_x v + 2\lambda_0 v = h_u(t) \\ v\big|_{t=0} = w\big|_{t=0} = u_0.u_0; \ v\big|_{\partial\Omega} = 0 \end{cases} \tag{2.4}$$

Due to the comparison principle (see for instance [14] or [21]),

$$w(t,x) \le v(t,x), \quad (t,x) \in [0,T] \times \Omega \tag{2.5}$$

Applying Proposition 1.3 to the linear equation (2.4) we obtain using Sobolev embedding theorem $W^{2-\delta,p} \subset C$ if $\delta < 2 - \frac{n}{p}$ that

$$\begin{aligned} |w(T,x_0)|^p \le |v(T,x_0)|^p \le C\|v(t)\|^p_{b,2-\delta,p} Ce^{-2\gamma T}\|v(0)\|^p_{b,2-\delta,p} + \\ + C\left(1 + \int_0^T e^{2\gamma(t-T)}\|g(t)\cdot u(t,x)\|^p_{b,0,p}dt\right) \end{aligned} \tag{2.6}$$

Estimating the last integral in (2.6) by Holder inequality and using the fact that the space $W^{2-\delta,p}$ is an algebra if $\delta < 2 - \frac{n}{p}$ we derive that

$$\|u(T)\|^p_{0,\infty} \le Ce^{-\gamma T}\|u(0)\|^p_{2-\delta,p} + \mu \sup_{t\in[0,T]} e^{\gamma(t-T)}\|u(t)\|^p_{0,\infty} + C_\mu\left(1 + \|g\|^p_{b,0,p}\right) \tag{2.7}$$

To complete the proof of Theorem 2.1 we need the following simple lemma

Lemma 2.1. *Let the function $Z(t)$ be a solution of the following inequality*

$$Z(T) \le C_1 e^{-\beta T} + C_2 + \mu \sup_{t\in[0,T]} \{e^{\beta(t-T)} Z(t)\} \tag{2.8}$$

and let $\mu \le 1/2$ and $\beta > 0$. Then

$$Z(T) \le 2C_1 e^{-\beta T} + 2C_2 \tag{2.9}$$

The proof of this lemma can be found, for instance in [9].

Applying the result of Lemma 2.1 to the estimate (2.7) with $Z(t) = \|u(t)\|^p_{0,\infty}$ and taking μ small enough we obtain the assertion of the theorem.

Theorem 2.2. *Let u be a solution of the problem* (0.1) *and let $\delta < \min\{2-r, 2-\frac{n}{p}\}$. Then the following estimate is valid:*

$$\|u(T)\|_{b,2-\delta,p} \le Q\left(\|u(0)\|_{b,2-\delta,p}\right) e^{-\gamma T} + Q\left(\|g\|_{b,0,p}\right) \tag{2.10}$$

Here $\gamma > 0$ and Q is a certain monotonic function independent of $u(0)$.

Proof. Let us rewrite the equation (0.1) in the form of linear one

$$\partial_t u = \Delta_x u - \lambda_0 u + h_f(t) \tag{2.11}$$

where $h_f(t) \equiv g - f(u(t), \nabla_x u(t))$. Applying the estimate (1.20) to the equation (2.11) we derive that

$$\|u(T)\|_{b,2-\delta,p} \le Ce^{-\gamma T}\|u(0)\|_{b,2-\delta,p} + C\|g\|_{b,0,p} + \\ + C\sup\{e^{\gamma(t-T)}\|f(u(t), \nabla_x u(t))\|_{b,0,p}\} \tag{2.12}$$

Estimating the last term in the right-hand side of (2.12) by using the assumptions (0.2) we obtain that

$$\|f(u(t), \nabla_x u(t))\|_{b,0,p} \le Q_1(\|u(t)\|_{0,\infty})\left(1 + \|u(t)\|^r_{b,1,rp}\right) \tag{2.13}$$

The interpolation inequality [19] implies that

$$\|u(t)\|_{b,1,p(2-\delta)} \le C\|u(t)\|_{0,\infty}^{1-\frac{1}{2-\delta}} \|u(t)\|_{b,2-\delta,p}^{\frac{1}{2-\delta}} \tag{2.14}$$

Since $\frac{r}{2-\delta} < 1$ then $\|u\|_{b,1,rp} \le C\|u\|_{b,1,p(2-\delta)}$ and it follows from (2.13) and (2.14) that for every $\mu > 0$ the following esimate is valid

$$\|f(u(t), \nabla_x u(t))\|_{b,0,p} \le Q_\mu(\|u(t)\|_{0,\infty}) + \mu\|u(t)\|_{b,2-\delta,p} \tag{2.15}$$

To complete the proof of Theorem 2.2 we need the following Lemma.

Lemma 2.2. *Let the function ϕ be continuous and the function u satisfy the estimate* (2.2)*. Then*

$$\|\phi(u(t))\|_{L^\infty(\Omega)} \le Q(\|u(0)\|_{b,2-\delta,p})e^{-\gamma T} + Q(\|g\|_{b,0,p}) \tag{2.16}$$

For a certain monotonic function Q.

The proof of Lemma 2.2 is given in [20].

Inserting the estimates (2.15) and (2.16) into the inequality (2.12) we obtain that

$$\|u(T)\|_{b,2-\delta,p} \le \widehat{Q}_\mu(\|u(0)\|_{b,2-\delta,p})e^{-\gamma T} + \widehat{Q}_\mu(\|g\|_{b,0,p}) + \\ + \mu\sup\{e^{\gamma(t-T)}\|u(t)\|_{b,2-\delta,p}\} \tag{2.17}$$

Applying now the assertion of Lemma 2.1 to (2.17) with μ small enough we obtain inequality (2.10) □

Remark 2.1. *Note that all estimates derived above depend only on the constant* K, R_0 *which are defined in the assumptions* (1.5) *and* (1.6)*. Thus if we consider a sequence* Ω_N *which satisfy these assumptions uniformly with respect to* $N \in \mathbb{N}$ *then the function* Q *in* (2.10) *can be chosen independently of* Ω_N*.*

Now we are in position to prove the existence of solutions for the problem (0.1). To this end we prove firstly this existence in the case when the domain Ω is bounded.

Theorem 2.3. *Let the above assumptions be valid and let* Ω *be bounded. Then the problem* (3.1) *has at least one solution in the space*

$$W_{\Omega}([0,T]) = C([0,T], W^{2-\delta,p}(\Omega)) \cap C^{1-\delta/2}([0,T], L^p(\Omega)) \tag{2.18}$$

and the following estimate is valid:

$$\|u\|_{W_{\Omega}([0,T])} \leq Q(\|u(0), \Omega\|_{b,2-\delta,p}) + Q(\|g\|_{b,0,p}) \tag{2.19}$$

Proof. A priori estimate (2.19) is an immediate corollary of the estimate (2.10) and the existence of solutions for the equation (0.1) can be deduced from this estimate in a standard way involving for instance Leray-Schauder principle (see [11] or [22]). Theorem 2.3 is proved.

Theorem 2.4. *Let the above assumptions hold and let* Ω *be an arbitrary unbounded domain which satisfies the assumptions* (1.5) *and* (1.6)*. Then the problem* (0.1) *has at least one solution from the class* (2.1)*.*

Proof. Let Ω_N, $N = 1, 2, \cdots$ be the sequence of smooth bounded domains, which satisfy the conditions (1.5) and (1.6) uniformly with respect to $N \in \mathbb{N}$, such that

$$\begin{cases} \Omega_N \subset \Omega_{N+1} \subset \Omega \ ; \ \Omega = \cup_{N=1}^{\infty} \Omega_N \\ \Omega \cap B_0^N \subset \Omega_N \subset \Omega \cap B_0^{N+1} \end{cases} \tag{2.20}$$

It is not difficult to check that such sequence exists.

Let us introduce the sequence of the cut–off functions $\psi_N(x) \in C_0^{\infty}(\mathbb{R}^n)$ such that $\psi_N(x) = 1$ if $x \in B_0^{N-1}$, $\psi_N(x) = 0$ if $x \notin B_0^N$ and $\|\psi_N\|_{C^2} \leq C$.

Let u_N be a solution of the following problem

$$\begin{cases} \partial_t u_N - \Delta_x u_N + \lambda_0 u_N + f(u_N, \nabla_x u_N) = g \\ u_N\big|_{\partial\Omega_N} = 0 \ ; \ u_N\big|_{t=0} = \psi_N u_0 \end{cases} \tag{2.21}$$

According to Remark 2.1 the estimates (2.10) with u replaced by u_N are valid uniformly with respect to $N \in \mathbb{N}$. Thus, for every $M \in \mathbb{N}$ the sequence $u_N\big|_{\Omega \cap B_0^M}$, $N \geq M$ is bounded in the space $W_{\Omega \cap B_0^M}([0,T])$. Using Cantor's diagonal procedure we can extract from the sequence u_N a subsequence (which we denote by u_N also for simplicity) converging $*$-weakly to u in $L^{\infty}([0,T], W^{2-\delta,p}(\Omega \cap B_0^M))$ for every $M \in \mathbb{N}$. It follows now from the standard reasonings (see, for example, [9]) that u is a solution of the equation (0.1). Theorem 2.4 is proved.

§3 The nonlinear equation. Uniqueness of solutions.

In this Section we prove the uniqueness problem for equation (0.1) and obtain a number useful estimates for the difference between two solutions of this equation

Theorem 3.1. *Let the assumptions of previous Section hold and let $u_1(t)$ and $u_2(t)$ be two solutions of the equation (0.1). Then the following esimate is valid*

$$\|u_1(t)-u_2(t),\Omega\cap B^1_{x_0}\|^2_{0,2}+\int_0^t\|\nabla_x u_1(s)-\nabla_x u_2(s),\Omega\cap B^1_{x_0}\|^2_{0,2}\,ds\le \tag{3.1}$$
$$\le C_1e^{C_2t}\int_{x\in\Omega}e^{-\varepsilon|x-x_0|}\|u_1(0)-u_2(0),\Omega\cap B^1_x\|^2_{0,2}$$

where the constants C_i depend on the initial values $\|u_i\|_{b,2-\delta,p}$ and $\varepsilon>0$ is small enough.

Particularly the solution of the problem (0.1), constructed in previous Section is unique.

Proof. Let u_1,u_2 be two solutions of (0.1) and $v(t)=u_2(t)-u_1(t)$. Then

$$\begin{cases}\partial_t v-\Delta_x v+\lambda_0 v=-\widehat{L}_1(t,x)v-\widehat{L}_2(t,x)\nabla_x v\\ v\big|_{t=0}=u_2(0)-u_1(0);\quad v\big|_{\partial\Omega}=0\end{cases}\tag{3.2}$$

Here

$$\begin{cases}\widehat{L}_1(t,x)=\int_0^1 f'_u(u_1+\theta v,\nabla_x u_1+\theta\nabla_x v)\,d\theta\\ \widehat{L}_2(t,x)=\int_0^1 f'_{\nabla_x u}(u_1+\theta v,\nabla_x u_1+\theta\nabla_x v)\,d\theta\end{cases}\tag{3.3}$$

It follows from the conditions (0.2) that

$$\begin{cases}|\widehat{L}_1(t)|\le Q(\|u(t)\|_{0,\infty})(1+|\nabla_x u(t)|^r)\\ |\widehat{L}_2(t)|\le Q(\|u(t)\|_{0,\infty})(1+|\nabla_x u(t)|^{r-1})\end{cases}\tag{3.4}$$

We denote here by $|\nabla_x u(t)|^r=|\nabla_x u_1(t)|^r+|\nabla_x u_2(t)|^r$.

After multiplying the equation (3.2) by $ve^{-\varepsilon|x-x_0|}$ in the space $L^2(\Omega)^k$ for sufficiently small $\varepsilon>0$ we obtain after simple transformations

$$\partial_t\left(|v(t)|^2,e^{-\varepsilon|x-x_0|}\right)+\left(|\nabla_x v(t)|^2,e^{-\varepsilon|x-x_0}\right)+\lambda_0\left(|v(t)|^2,e^{-\varepsilon|x-x_0|}\right)+ \tag{3.5}$$
$$+\left(\widehat{L}_1 v,ve^{-\varepsilon|x-x_0|}\right)+\left(\widehat{L}_2\nabla_x v,ve^{-\varepsilon|x-x_0|}\right)\le 0$$

Let's estimate the two nonlinear terms in (3.5) separately.

It follows from the estimate (2.2) that $|u_i(t,x)|\le C$ for all $t\in[0,T]$, $x\in\Omega$. Hence

(3.6)
$$I_1(t)\ \equiv\ |\left(L_1\,v,ve^{-\varepsilon|x-x_0|}\right)|\ \le\ C_1\left(|v(t)|^2,e^{-\varepsilon|x-x_0|}\right)\ +\ C_1\left(|\nabla_x u|^r v,ve^{-\varepsilon|x-x_0|}\right)$$

Let us estimate the last integral at the right–hand side of (3.6). To this end we use a trick based on (1.10), Holder inequality, and embedding Theorem $W^{1-\theta,2}\subset L^q$ where $\frac{1}{q}=\frac{1}{2}-\frac{1-\theta}{n}$ and $\theta>0$ is small enough and satisfies the equation

$$\frac{r}{p(2-\delta)}+1-\frac{2(1-\theta)}{n}=1,\ \Leftrightarrow\ 1-\theta=\frac{n}{2p}\cdot\frac{r}{2-\delta}<1$$

Indeed,

$$(3.7)\quad \left(|\nabla_x u|^r v, ve^{-\varepsilon|x-x_0|}\right) \le C\int_\Omega e^{-\varepsilon|x_0|}\|v\cdot v\cdot|\nabla_x u|^r, V_{x_0}\|_{0,1}\,dx_0 \le$$
$$\le C_1\|u,\Omega\|^r_{b,1,p(2-\delta)}\int_\Omega e^{-\varepsilon|x_0|}\|v, V_{x_0}\|^2_{0,q}\,dx_0 \le$$
$$\le C_2\|u,\Omega\|^r_{b,1,p(2-\delta)}\int_\Omega e^{-\varepsilon|x_0|}\|v, V_{x_0}\|^2_{1-\theta,2}\,dx_0$$

Using the inequality (2.14), interpolation inequality

$$(3.8)\qquad \|v\|_{1-\theta,2} \le C\|v\|^\theta_{0,2}\|v\|^\theta_{1,2} \le C_\mu\|v\|^2_{0,2} + \mu\|v\|^2_{1,2}$$

and the estimate (2.10) we finally obtain that

$$(3.9)\qquad I_1(t) \le C_\mu\left(|v(t)|^2, e^{-\varepsilon|x-x_0|}\right) + \mu\left(|\nabla_x v(t)|^2, e^{-\varepsilon|x-x_0|}\right)$$

Analogously

$$(3.10)\quad I_2(t) \equiv \left(\widehat{L}_2\nabla_x v, ve^{-\varepsilon|x-x_0|}\right) \le C\left(|v|, |\nabla_x v|e^{-\varepsilon|x-x_0|}\right) +$$
$$+ C\left(|\nabla_x u|^{r-1}|\nabla_x v|, |v|e^{-\varepsilon|x-x_0|}\right)$$

Arguing as in (3.7) and using Holder inequality with exponents $\frac{p(2-\delta)}{r-1}$, 2 and q where $\frac{1}{q} = \frac{1}{2} - \frac{1-\theta}{n}$ and the exponent θ can be found from the equation

$$\frac{r-1}{p(2-\delta)} + \frac{1}{2} + \frac{1}{2} - \frac{1-\theta}{n} = 1, \;\Leftrightarrow\; 1-\theta = \frac{n}{2p}\cdot\frac{2(r-1)}{2-\delta} < \frac{n}{2p}\cdot\frac{r}{2-\delta} < 1$$

we obtain that

(3.11)
$$\left(|\nabla_x u|^{r-1}v, \nabla_x ve^{-\varepsilon|x-x_0|}\right) \le C\int_\Omega e^{-\varepsilon|x-x_0|}\|\,|v|\cdot|\nabla_x v|\cdot|\nabla_x u|^{r-1}, V_x\|_{0,1}\,dx \le$$
$$\le C_1\|u,\Omega\|^{r-1}_{b,1,p(2-\delta)}\int_\Omega e^{-\varepsilon|x-x_0|}\|\nabla_x v, V_x\|_{0,2}\|v, V_x\|_{0,q}\,dx \le$$
$$\le C_2\int_\Omega e^{-\varepsilon|x-x_0|}\|\nabla_x v, V_x\|_{0,2}\|v, V_x\|_{1-\theta,2}\,dx$$

Using now the interpolation inequality (3.8) we finally get

$$(3.12)\qquad I_2(t) \le C_\mu\left(|v(t)|^2, e^{-\varepsilon|x-x_0|}\right) + \mu\left(|\nabla_x v(t)|^2, e^{-\varepsilon|x-x_0|}\right)$$

Replace the integrals I_1 and I_2 in (3.5) by their estimates (3.10) and (3.12).

$$(3.13)\qquad \partial_t\left(|v(t)|^2, e^{-\varepsilon|x-x_0|}\right) + \beta\left(|\nabla_x v(t)|^2, e^{-\varepsilon|x-x_0|}\right) \le C\left(|v(t)|^2, e^{-\varepsilon|x-x_0|}\right)$$

Applying the Gronwall inequality to (3.13) we obtain the assertion of the theorem □

Theorem 3.2. *Let the assumptions of previous Theorem hold. Then the following esimate is valid*

$$\|u_1(t)-u_2(t),\Omega\cap B^1_{x_0}\|^2_{1,2}+\int_0^t\|u_1(s)-u_2(s),\Omega\cap B^1_{x_0}\|^2_{2,2}\,ds\le \tag{3.14}$$
$$\le C_1e^{C_2t}\int_{x\in\Omega}e^{-\varepsilon|x-x_0|}\|u_1(0)-u_2(0),\Omega\cap B^1_x\|^2_{1,2}dx$$

where the constants C_i depend on the initial values $\|u_i\|_{b,2-\delta,p}$ and $\varepsilon>0$ is small enough.

Proof. Applying the L^2_ε-parabolic regularity theorem (see, for instance, [9]) to the equation (3.2) we obtain that

(3.15)
$$\|v(T),\Omega\cap B^1_{x_0}\|^2_{1,2}+\int_0^T\|v(t),\Omega\cap B^1_{x_0}\|^2_{2,2}dt\le C\left(|v(0)|^2+|\nabla_xv(0)|^2,e^{-\varepsilon|x-x_0|}\right)e^{-\alpha t}+$$
$$+C\int_0^T\left(|\widehat{L}_1(t)v(t)|^2,e^{-\varepsilon|x-x_0|}\right)+\left(|\widehat{L}_2(t)\nabla_xv(t)|^2,e^{-\varepsilon|x-x_0|}\right)dt$$

So it remains to esimate the integrals into the right-hand side of the estimate (3.15).

Arguing as in the proof of previous Theorem we deduce the estimate

$$\left(|\widehat{L}_1(t)v(t)|^2,e^{-\varepsilon|x-x_0|}\right)+\left(|\widehat{L}_2(t)\nabla_xv(t)|^2,e^{-\varepsilon|x-x_0|}\right)\le \tag{3.16}$$
$$\le C_\mu\int_{x\in\Omega}e^{-\varepsilon|x-x_0|}\|v(t),\Omega\cap B^1_x\|^2_{1,2}dx+\mu\int_{x\in\Omega}e^{-\varepsilon|x-x_0|}\|v(t),\Omega\cap B^1_x\|^2_{2,2}dx$$

Inserting this estimate to the inequality (3.15) and using the estimate (3.1) we will have

$$\|v(T),\Omega\cap B^1_{x_0}\|^2_{1,2}+\int_0^T\|v(t),\Omega\cap B^1_{x_0}\|^2_{2,2}dt\le \tag{3.17}$$
$$\le C_\mu e^{Ct}\int_{x\in\Omega}e^{-\varepsilon|x-x_0|}\|v(0),\Omega\cap B^1_x\|^2_{1,2}dx+\mu\int_0^T\int_{x\in\Omega}e^{-\varepsilon|x-x_0|}\|v(t),\Omega\cap B^1_x\|^2_{2,2}dxdt$$

Multiplying (3.17) by $e^{-\varepsilon_1|z-x_0|}$, $\varepsilon_1<\varepsilon$, integrating over $x_0\in\Omega$, using (1.15) and taking μ small enough we obtain

(3.18)
$$\int_{x_0\in\Omega}e^{-\varepsilon_1|x_0-z|}\|v(T),\Omega\cap B^1_{x_0}\|^2_{1,2}dx_0+\int_0^T\int_{x_0\in\Omega}e^{-\varepsilon|z-x_0|}\|v(t),\Omega\cap B^1_{x_0}\|^2_{2,2}dx_0dt\le$$
$$\le Ce^{Ct}\int_{x_0\in\Omega}e^{-\varepsilon|z-x_0|}\|v(0),\Omega\cap B^1_{x_0}\|^2_{1,2}dx_0$$

Theorem 3.2 is proved.

In conclusion of this Section we consider a smoothing property for the equation (3.2).

Theorem 3.3. *Let the conditions of Theorem 3.1 hold. Then the following estimate is valid*

$$\|u_1(t) - u_2(t), \Omega\cap B^1_{x_0}\|_{1,2} \le \frac{C}{t}e^{C_1 t}\left(|u_1(0) - u_2(0)|^2, e^{-\varepsilon|x-x_0|}\right) \tag{3.19}$$

where constants C *and* C_1 *depends on* $\|u_i(0)\|_{b,2-\delta,p}$ *and* $\varepsilon > 0$.

Proof. Let $w(t) = tv(t)$. Then

$$\begin{cases} \partial_t w - \Delta_x w + \lambda_0 w = -\widehat{L}_1(t,x)w - \widehat{L}_2(t,x)\nabla_x w + v \\ w\big|_{t=0} = 0; \quad w\big|_{\partial\Omega} = 0 \end{cases} \tag{3.20}$$

Arguing as in the proof of Theorem 3.2 we derive that

$$\|w(T), \Omega\cap B^1_{x_0}\|_{1,2} \le Ce^{C_1 T}\int_0^T \left(|v(t)|^2, e^{-\varepsilon|x-x_0|}\right) dt \tag{3.21}$$

Estimating the right-hand side of (3.21) by (3.1) we obtain the assertion of the theorem.

Arguing analogously (see also [9], [24]) one can derive the following improved version of smoothing property for the equation (3.1)

Theorem 3.4. *Let the conditions of Theorem 3.1 hold. Then the following estimate is valid*

$$\|u_1(t) - u_2(t), \Omega\cap B^1_{x_0}\|^p_{2-\delta,p} \le \frac{C}{t^N}e^{C_1 t}\left(|u_1(0) - u_2(0)|^2, e^{-\varepsilon|x-x_0|}\right)^{p/2} \tag{3.22}$$

where constants C *and* C_1 *depends on* $\|u_i(0)\|_{b,2-\delta,p}$, $\varepsilon > 0$ *and* N *is large enough.*

Corollary 3.1. *Let* $u_1(t)$ *and* $u_2(t)$ *be two solutions of the equation* (0.1) *and let* $\phi \in L^1(\mathbb{R}^n)$ *be a weight function which satisfied Definition 1.1. Then the following estimates are valid*

$$\|u_1(1) - u_2(1)\|_{b,\phi,1,2} \le C_1\|u_1(0) - u_2(0)\|_{b,\phi,0,2} \tag{3.23}$$

and

$$\|u_1(1) - u_2(1)\|_{b,\phi^{p/2},2-\delta,p} \le C_2\|u_1(0) - u_2(0)\|_{b,\phi,0,2} \tag{3.24}$$

where the constants C_1 *and* C_2 *depends on* $\|u_i\|_{b,2-\delta,p}$ *and* C_ϕ *which is defined by* (1.1).

Indeed, the estimates (3.23) and (3.24) are immedate corollaries of the inequalities (3.19) and (3.22) and (1.17) (see also [24]).

§4 The attractor for the nonlinear equation.

In this Section we prove the existence of the attractor for the equation (3.1) in the space $W_b^{2-\delta,p}(\Omega)$.

Note that Theorems 2.1 and 3.1 imply that under the conditions of Section 2 the equation (0.1) generates a semigroup $S_t : \Phi_b(\Omega) \to \Phi_b(\Omega)$, where $\Phi_b(\Omega) = W_b^{2-\delta,p}(\Omega) \cap \{u_0\big|_{\partial\Omega} = 0\}$, by formula

$$S_t u(0) = u(t) \quad \text{where } u(t) \text{ is a solution of } (0.1) \tag{4.1}$$

Moreover, it follows from the estimate (2.10) that this semigroup possesses a bounded absorbing set K in the space $\Phi_b(\Omega)$, i.e. for any other bounded subset $B \subset \Phi_b(\Omega)$ there exists $T = T(B)$ such that

$$S_t B \subset K \text{ if } t \geq T$$

It seems natural to consider the attractor of (4.1) in the 'uniform' topology of the space $\Phi_b(\Omega)$ but, as it shown for instance in [24], in contrast to the case of bounded domains Ω in our situation the existence of a compact attractor for (4.1) in the 'uniform' topology of $\Phi_b(\Omega)$ is very restrictive assumption which violates even in the simplest examples. Indeed, as it shown in [24] the attractor $\mathcal{A}$ of the Chafee-Infante equation

$$\partial_t u = \Delta_x u - u^3 + u$$

in the whole space $\Omega = \mathbb{R}^n$ is not compact in Φ_b.

That's why we will construct below the attractor $\mathcal{A}$ of the semigroup (4.1) which attracts bounded subsets of $\Phi_b(\Omega)$ only in a local topology of the space $\Phi_{loc} = W^{2-\delta,p}_{loc}(\Omega)$ (i.e., $\mathcal{A}$ is the (Φ_b, Φ_{loc})-attractor of (4.1) in notations of [2]).

Recall that the space $\Phi_{loc}(\Omega)$ is reflexive metrizable F-space which defines by semi-norms $\| \cdot, \Omega \cap B^1_{x_0} \|_{2-\delta,p}$, $x_0 \in \Omega$.

Definition 4.1. *The set $\mathcal{A} \subset \Phi_b(\Omega)$ is defined to be the attractor of the semigroup S_t if the following assumptions hold:*

1. The set $\mathcal{A}$ is compact in $\Phi_{loc}(\Omega)$.

2 The set $\mathcal{A}$ is strictly invariant with respect to S_t, i.e.

$$S_t \mathcal{A} = \mathcal{A} \text{ for } t \geq 0$$

3. The set $\mathcal{A}$ is the attracting set for S_t in local topology, i.e. for every neighborhood $\mathcal{O}(\mathcal{A})$ of $\mathcal{A}$ in the topology of the space $\Phi_{loc}(\Omega)$ and for every bounded in uniform topology subset $B \subset \Phi_b(\Omega)$ there exists $T = T(\mathcal{O}, B)$ such that

$$S_t B \subset \mathcal{O}(\mathcal{A}) \text{ if } t \geq T$$

Recall that the first condition means that the restriction $\mathcal{A}\big|_{\Omega_1}$ is compact in the space $W^{2(1-1/p),p}(\Omega_1)$ for every *bounded* $\Omega_1 \subset \Omega$.

Analogously, the third condition means that for every *bounded* $\Omega_1 \subset \Omega$, every bounded B in $\Phi_b(\Omega)$ and every $W^{2-\delta,p}(\Omega_1)$-neighborhood $\mathcal{O}(\mathcal{A}\big|_{\Omega_1})$ of the restriction $\mathcal{A}\big|_{\Omega_1}$ there exists $T = T(\Omega_1, \mathcal{O}, B)$ such that

$$(S_t B)\big|_{\Omega_1} \subset \mathcal{O}(\mathcal{A}\big|_{\Omega_1}) \text{ if } t \geq T$$

Theorem 4.1. *Let the above assumptions be valid. Then the semigroup S_t, defined by* (4.1)*, possesses an attractor $\mathcal{A}$ in the sense of Definition 4.1 which has the following structure:*

$$\mathcal{A} = \mathcal{K}\big|_{t=0} \tag{4.2}$$

where we denote by $\mathcal{K}$ the set of all solutions of (0.1)*, defined and bounded for all $t \in \mathbb{R}$ ($\sup_{t \in \mathbb{R}} \|u(t)\|_{\Phi_b(\Omega)} < \infty$).*

Proof. According to the attractor's existence theorem for abstract semigroups (see [2]), it is sufficient to verify the following conditions:

1. The operators S_t is Φ_{loc}-continuous on every Φ_b-bounded set and for every fixed $t \geq 0$.

2. The semigroup S_t possesses the precompact attracting set in Φ_{loc}-topology.

The continuity of S_t is an immediate corollary of Theorem 3.4. Since it remains only to construct the compact attracting set.

According to Theorem 2.2, the set $B_R = \{u_0 \in \Phi_b : \|u_0\|_{\Phi_b} \leq R\}$ is the absorbing set for the semigroup S_t if R is large enough. Hence the set $K = S_1 B_R$ is the absorbing set also. So it remains to prove that K is precompact in Φ_{loc}.

According to Cantor's diagonal procedure it is sufficient to prove that the restriction $K\big|_{\Omega_1}$ is precompact for every bounded $\Omega_1 \subset \Omega$. To this end we fix an arbitrary bounded subdomain $\Omega_1 \subset \Omega$ and consider an arbitrary sequence $u_n(1)$, $n \in \mathbb{N}$, $u_n(0) \in B_R$.

Since Ω_1 is bounded that $\Omega_1 \subset B_0^M$ for a sufficiently large M. Let $\psi(x)$ be the cut-off function, such that $\psi(x) = 1$ if $|x| \leq M$ and $\psi(x) = 0$ if $|x| > M + 1$. Then $\psi\big|_{\Omega_1} \equiv 1$. Let us consider now a sequence $w_n(t) = t\psi(x)u_n(t)$ which evidently satisfy the equations

$$\begin{cases} \partial_t w_n(t) - \Delta_x w_n(t) + \lambda_0 w_n(t) = -t\psi(x)f(u_n(t), \nabla_x u_n(t)) + \\ +t\psi(x)g - t\Delta_x\psi(x)u_n(t) - 2t\nabla_x\psi(x).\nabla_x u_n(t) + \psi(x)u_n(t) \equiv h_n(t) \\ w_n(0) = 0; \quad w_n\big|_{\tilde{\Omega}_M} = 0 \end{cases} \tag{4.3}$$

where $\tilde{\Omega}_M \subset \Omega$ is bounded domain with smooth boundary such that $\Omega \cap B_0^{M+1} \subset \tilde{\Omega}_M$.

According to Corollary 3.2, the sequence $u_n(t)$ is bounded in

$$W_M = C([0,1], W^{2-\delta,p}(\tilde{\Omega}_M)) \cup C^{1-\delta/2}([0,1], L^p(\tilde{\Omega}_M)) \tag{4.4}$$

hence without loss of generality we may assume that $u_n \to u$ $*$-weakly in the space $L^\infty([0,1], W^{2-\delta,p})$).

Using the compactness of embedding $W_M \subset C([0,1], W^{1,rp}(\tilde{\Omega}_M)) \cap C([0,1] \times \tilde{\Omega}_M)$ one can easily derive that $h_n \to h$ strongly in $C([0,1] \times \tilde{\Omega}_M)$. The parabolic regularity theorem, applied to the equation (4.4), implies now that $u_n(1) \to u(1)$ in $W^{2-\delta,p}(\Omega_1)$. Theorem 4.1 is proved.

§5 Kolmogorov's ε-entropy: definitions and typical examples.

In this Section we recall briefly the definition of ε-entropy and give the upper and lower estimates of it when $\varepsilon \to 0$ for the typical sets in functional spaces. For the detailed study of this concept see [13], [19].

Definition 5.1. *Let $\mathbb{M}$ be a metric space and let K be precompact subset of it. For a given $\varepsilon > 0$ let $N_\varepsilon(K) = N_\varepsilon(K, \mathbb{M})$ be the minimal number of ε-balls in $\mathbb{M}$ which cover the set K (this number is evidently finite by Hausdorff criteria). By definition, Kolmogorov's ε-entropy of K in $\mathbb{M}$ is the following number*

$$\mathbb{H}_\varepsilon(K) = \mathbb{H}_\varepsilon(K, \mathbb{M}) \equiv \ln N_\varepsilon(K) \tag{5.1}$$

Example 5.1. Let K be compact n-dimensional Lipschitz manifold in $\mathbb{M}$. Then the evident estimates imply that

$$C_1 \left(\frac{1}{\varepsilon}\right)^n \leq N_\varepsilon(K) \leq C_2 \left(\frac{1}{\varepsilon}\right)^n \tag{5.2}$$

and consequently

$$\mathbb{H}_\varepsilon(K) = (n + \bar{\bar{o}}(1)) \ln \frac{1}{\varepsilon} \tag{5.3}$$

when $\varepsilon \to 0$.

This example justifies the following definition

Definition 5.2. *The fractal (box-counting) dimension of the set $K \subset\subset \mathbb{M}$ is defined to be the following number:*

$$\dim_F(K) = \dim_F(K, \mathbb{M}) = \limsup_{\varepsilon \to 0} \frac{\mathbb{H}_\varepsilon(K)}{\ln \frac{1}{\varepsilon}} \tag{5.4}$$

Note that the fractal dimension $\dim_F(K) \in [0, \infty]$ is defined for any compact set in $\mathbb{M}$ but may be not integer if K is not a manifold.

Example 5.2. Let $\mathbb{M} = [0, 1]$ and let K be the ternary Cantor set in $\mathbb{M}$. Then it is not difficult to obtain that

$$C_1 \left(\frac{1}{\varepsilon}\right)^d \le N_\varepsilon(K) \le C_2 \left(\frac{1}{\varepsilon}\right)^d \quad , \quad d = \frac{\ln 2}{\ln 3} \tag{5.5}$$

and consequently $\dim_F(K) = d = \frac{\ln 2}{\ln 3}$.

Consider now the examples of infinite dimensional sets (i.e. $\dim_F(K) = \infty$).

The following two examples give the typical asymptotics for the entropy in the spaces of analitycal functions.

Example 5.3. Let K be the set of all analytic functions f in a ball $B(R)$ of radius $R > 1$ in $\mathbb{C}^n$ such that $\|f\|_{L^\infty(B(R))} \le 1$ and let $\mathbb{M}$ be the space $C(B^{Re})$, where $B^{Re} = \{z \in \mathbb{C}^n \ : \operatorname{Im} z_i = 0\,, |z| \le 1\}$. Thus, K consists of all functions from $C(B^{Re})$ which can be extended holomorphically to the ball $B(R) \subset \mathbb{C}^n$ and the C-norm of this extension is not greater then one. Then

$$C_1 \left(\ln \frac{1}{\varepsilon}\right)^{n+1} \le \mathbb{H}_\varepsilon(K, \mathbb{M}) \le C_2 \left(\ln \frac{1}{\varepsilon}\right)^{n+1} \tag{5.6}$$

For the proof of this estimate see [13].

Example 5.4. Let $\mathbb{M}$ be the same as in previous example and let K be the set of all functions f in $\mathbb{M}$ which can be extended to the entire function $\widehat{f}$ in $\mathbb{C}^n$ which satisfy the estimate

$$|\widehat{f}(z)| \le K_1 e^{K_2|z|}, \quad z \in \mathbb{C}^n \tag{5.7}$$

Then, as proved in [13],

$$C_1 \frac{\left(\ln \frac{1}{\varepsilon}\right)^{n+1}}{\left(\ln \ln \frac{1}{\varepsilon}\right)^n} \le \mathbb{H}_\varepsilon(K) \le C_2 \frac{\left(\ln \frac{1}{\varepsilon}\right)^{n+1}}{\left(\ln \ln \frac{1}{\varepsilon}\right)^n} \tag{5.8}$$

The next example gives the typical asymptotics for the entropy in the class of Sobolev spaces in bounded domains.

Example 5.5. Let Ω be smooth bounded domain in $\mathbb{R}^n$ and

$$W^{l_1,p_1}(\Omega) \subset\subset W^{l_2,p_2}(\Omega)\ ,\ 0 \leq l_i < \infty,\ 1 < p_i < \infty,\ l_1 > l_2$$

i.e., according to the embedding theorem $\frac{l_1}{n} - \frac{1}{p_1} > \frac{l_2}{n} - \frac{1}{p_2}$.

Let now $\mathbb{M} = W^{l_2,p_2}(\Omega)$ and K be the unitary ball in $W^{l_1,p_1}(\Omega)$. Then

$$C_1\left(\frac{1}{\varepsilon}\right)^{\frac{n}{l_1-l_2}} \leq \mathbb{H}_\varepsilon(K) \leq C_2\left(\frac{1}{\varepsilon}\right)^{\frac{n}{l_1-l_2}} \tag{5.9}$$

The proof of this estimate can be found in [19].

The following class of functions will be essentially used in the next Section in order to obtain the lower bounds of ε-entropy of attractors.

Definition 5.3. *Let us denote by $\mathbb{B}_\sigma(\mathbb{R}^n)$ the subspace of $L^\infty(\mathbb{R}^n)$ which consists of all functions ϕ with the Fourier transform $\widehat{\phi}$ satisfying the condition*

$$\operatorname{supp}\widehat{\phi} \subset [-\sigma,\sigma]^n \tag{5.10}$$

It is well-known that every function $\phi \in \mathbb{B}_\sigma$ can be extended to entire function $\tilde{\phi}(z) \in A(\mathbb{C}^n)$ which satisfy the estimate

$$\sup_{x\in\mathbb{R}^n} |\tilde{\phi}(x+iy)| \leq C\|\phi,\mathbb{R}^n\|_{0,\infty} e^{\sigma\sum_{i=1}^n |y_i|} \tag{5.11}$$

Moreover, every function $\phi \in L^\infty$, which possesses the entire extension $\tilde{\phi}$ which satisfies (5.11) belongs in fact to the space $\mathbb{B}_\sigma$.

Example 5.6. Let $K = B(0,1,\mathbb{B}_\sigma)$, $\mathbb{M} = C(B_0^R)$. Then

$$\mathbb{H}_\varepsilon(B(0,1,\mathbb{B}_\sigma), C_b(B_0^R)) \leq C(R + K\ln\frac{1}{\varepsilon})^n \ln\frac{1}{\varepsilon} \tag{5.12}$$

Moreover C and K are independent of R.

For the proof of this estimate see for instance [24]. We formulate in conclusion the lower bounds for the entropy form Example 5= =2E6.

Proposition 5.1. *The following estimate is valid for $R \geq R_0$ and $\varepsilon < \varepsilon_0$*

$$\mathbb{H}_\varepsilon\left(B(0,1,\mathbb{B}_\sigma), C_b(B_0^R)\right) \geq CR^n \ln\frac{1}{\varepsilon} \tag{5.13}$$

where the constant C is independent of R and ε.

For the proof of (5.13) see for instance [13]. Thus, the estimate (5.12) is sharp for $R \sim \ln\frac{1}{\varepsilon}$ and $R >> \ln\frac{1}{\varepsilon}$. For the case $R << \ln\frac{1}{\varepsilon}$ we formulate only the following result (see [24]).

Proposition 5.2. *For every $\delta > 0$ there exists $C_\delta > 0$ such that*

$$\mathbb{H}_\varepsilon\left(B(0,1,\mathbb{B}_\sigma), C(B_0^1)\right) \geq C_\delta\left(\ln\frac{1}{\varepsilon}\right)^{n+1-\delta} \tag{5.14}$$

And consequently, the estimate (5.12) is sharp for the case $R << \ln\frac{1}{\varepsilon}$ also.

Remark 5.1. Instead of the spaces $\mathbb{B}_\sigma$ one can consider a slighty general class $\mathbb{B}_{\sigma,\xi}$, $\xi \subset \mathbb{R}^k$ which consists of functions ϕ with Fourier transform $\widehat{\phi}$ satisfying the assumption

$$\operatorname{supp}\widehat{\phi} \subset \xi + [-\sigma,\sigma]^n \tag{5.15}$$

Then the estimates (5.12), (5.13) and (5.14) are evidently valid for the class $\mathbb{B}_{\sigma,\xi}$ also.

§6 The entropy of the attractor: the upper bounds.

In this Section using the tecnique developed in [24] we obtain the upper estimates of ε-entropy for the attractor $\mathcal{A}$ of the equation (0.1). Recall that we construct the attractor $\mathcal{A}$ which is compact only in F-space Φ_{loc} but not in the uniform topology of $\Phi_b(\Omega)$. That's why we will estimate the entropy of the restrictions $\mathcal{A}\big|_{\Omega\cap B^R_{x_0}}$.

Theorem 6.1. *Let the assumptions of Section 2 be valid and let*

$$\mathrm{vol}_{\Omega,x_0}(R) = \mathrm{vol}(\Omega\cap B^R_{x_0}) \tag{6.1}$$

Then for every $R\in\mathbb{R}_+$, $x_0\in\Omega$

$$\mathbb{H}_\varepsilon\left(\mathcal{A}\big|_{\Omega\cap B^R_{x_0}}, W^{2-\delta,p}_b(\Omega\cap B^R_{x_0})\right) \le C\,\mathrm{vol}_{\Omega,x_0}(R+K\ln\frac{1}{\varepsilon})\ln\frac{1}{\varepsilon} \tag{6.2}$$

where the constants C, K and L are independent of R and $x_0\in\Omega$.

The proof of this Theorem is based on the estimates (3.23) and (3.24) with a special choice of the weight function ϕ and comletely analogous to the proof of [24,Th. 8.1]. Indeed, define a family of weight functions with the rate of growth 1 by the following formula

$$\phi_{R,x_0}(x) = \begin{cases} e^{R-|x-x_0|} & \text{if } |x-x_0|\ge R \\ =09=091 & \text{if } |x-x_0|\le R=09=09 \end{cases} \tag{6.3}$$

It follows from the definition of these functions that

$$\mathbb{H}_\varepsilon\left(\mathcal{A}\big|_{\Omega\cap B^R_{x_0}}, W^{2-\delta,p}_b(\Omega\cap B^R_{x_0})\right) \le \mathbb{H}_\varepsilon\left(\mathcal{A}, W^{2-\delta,p}_{b,\phi_{R,x_0}}(\Omega)\right) \tag{6.4}$$

Hence, instead of estimating the entropy of the restriction $\mathcal{A}\big|_{\Omega\cap B^R_{x_0}}$ it is sufficient to estimate the entropy of the attractor in weighted Sobolev spaces $W^{2-\delta,p}_{b,\phi_{R,x_0}}(\Omega)$.

Let now $u_1(t)$ and $u_2(t)$ be two solutions of the equation (0.1) which lie on the attractor $\mathcal{A}$. Then, according to the estimates (3.24)

$$\|u_1(1)-u_2(1)\|_{W^{2-\delta,p}_{b,\phi^{p/2}_{R,x_0}}(\Omega)} \le C\|u_1(0)-u_2(0)\|_{L^2_{b,\phi_{R,x_0}}(\Omega)} \tag{6.5}$$

Here the constant C in (6.5) is independent of $u_1,u_2\in\mathcal{A}$. Moreover, since

$$\phi_{R,x_0}(x+y)\le e^{|x|}\phi_{R,x_0}(y)$$

then $C_{\phi_{R,x_0}}\equiv 1$ and consequently C is independent of R and x_0 also.

The estimate (6.5) together with the description (4.2) of the attractor $\mathcal{A}$ implies immediately that

$$\mathbb{H}_\varepsilon\left(\mathcal{A}, W^{2-\delta,p}_{b,\phi^{p/2}_{R,x_0}}(\Omega)\right) \le \mathbb{H}_{\varepsilon/(2C)}\left(\mathcal{A}, L^2_{b,\phi_{R,x_0}}(\Omega)\right) \tag{6.6}$$

The estimate (6.6) reduces our problem to estimating the entropy of the attractor in the space $L^2_{b,\phi_{R,x_0}}(\Omega)$.

The following corollary of the estimate (3.23) is of fundamental significance for this estimation:

Let u_1 and u_2 be arbitrary two solutions of the equation (0.1) which belong to the attractor. Then the following estimate is valid

$$\|u_1(1)-u_2(1)\|_{W^{1,2}_{b,\phi_{R,x_0}}(\Omega)} \le C\|u_1(0)-u_2(0)\|_{L^2_{b,\phi_{R,x_0}}(\Omega)} \tag{6.7}$$

Where the constant C depends only on the equation. Indeed it has been proved in [24] that (6.7) implies the following reccurent estimate

Lemma 6.1. *Let* (6.7) *be valid. Then*

$$\mathbb{H}_{\varepsilon/2^k}\left(\mathcal{A}, L^2_{b,\phi_{R,x_0}}\right) \le \mathbb{H}_{\varepsilon}\left(\mathcal{A}, L^2_{b,\phi_{R,x_0}}\right) + k\ln M_k(\varepsilon) \tag{6.8}$$

where

$$\ln M_k(\varepsilon) \le C\,\mathrm{vol}_{\Omega,x_0}(R + L\ln\frac{2^k}{\varepsilon}) \tag{6.9}$$

Moreover, the constants C and L is independent of k, R, ε and x_0.

The estimate (6.2) is an immediate corollary of (6.8). Indeed since $\mathcal{A}$ is bounded in Φ_b then there exists $R_0 > 0$, such that $\mathbb{H}_{R_0}(\mathcal{A}, L^2_{b,\phi_{R,x_0}}) = 0$ for every R and x_0. The estimate (6.8) implies now that

$$\mathbb{H}_{R_0/2^k}\left(\mathcal{A}, L^2_{b,\phi_{R,x_0}}\right) \le +Ck\,\mathrm{vol}_{\Omega,x_0}(R + L\ln\frac{2^k}{R_0}) \tag{6.10}$$

Fixing now $k \sim \ln\frac{R_0}{\varepsilon}$ we obtain (6.2). Theorem 6.1 is proved.

We consider now a number of corollaries of Theorem 6.1.

Corollary 6.1. Since $C \subset W^{2-\delta,p}_b$ then

$$\mathbb{H}_{\varepsilon}\left(\mathcal{A}, C(\Omega\cap B^R_{x_0})\right) \le C\,\mathrm{vol}_{\Omega,x_0}(R + K\ln\frac{1}{\varepsilon})\ln\frac{1}{\varepsilon} \tag{6.11}$$

Corollary 6.2. Let $\Omega = \mathbb{R}^n$. Then $\mathrm{vol}_{\Omega,x_0}(r) = cr^n$ and consequently

$$\mathbb{H}_{\varepsilon}\left(\mathcal{A}, W^{2-\delta,p}_b(B^R_{x_0})\right) \le C\left(R + K\ln\frac{1}{\varepsilon}\right)^n \ln\frac{1}{\varepsilon} \tag{6.12}$$

Taking $R = \ln\frac{1}{\varepsilon}$ we obtain that

$$\mathbb{H}_{\varepsilon}\left(\mathcal{A}, W^{2-\delta,p}_b(B^{\ln\frac{1}{\varepsilon}}_{x_0})\right) \le C_1\left(\ln\frac{1}{\varepsilon}\right)^{n+1} \tag{6.13}$$

Note that the estimate (6.12) gives the same type of upper bounds for $R = 1$ and $R = \ln\frac{1}{\varepsilon}$.

Corollary 6.3. Let Ω be a bounded domain. Then Theorem 6 implies the estimate

$$\mathbb{H}_{\varepsilon}\left(\mathcal{A}, W^{2-\delta,p}_b(\Omega)\right) \le C\,\mathrm{vol}(\Omega)\ln\frac{1}{\varepsilon} \tag{6.14}$$

which reflects the well-known fact that in this case the attractor $\mathcal{A}$ has the finite fractal dimension. Note however that even for bounded domains this estimate (6.14) for the case where f depends explicitly on a gradient $\nabla_x u$ is of independent inerest.

Corollary 6.4. Let $\Omega = \mathbb{R}^k \times \omega^{n-k}$ be a cylindrical domain where ω is bounded. Then the estimate (6.1) gives the following bound of the ε-entropy of the autonomous attractor

$$\mathbb{H}_{\varepsilon}\left(\mathcal{A}, W^{2-\delta,p}_b(\Omega\cap B^R_{x_0})\right) \le C\left(R + K\ln\frac{1}{\varepsilon}\right)^k \ln\frac{1}{\varepsilon} \tag{6.15}$$

Definition 6.1. *Let $\mathcal{A} \subset \Phi_b(\Omega)$ be a compact set in the space $\Phi_{loc}(\Omega)$. Then the ε-entropy per unit volume is defined to be the following number*

$$\overline{\mathbb{H}}_\varepsilon(\mathcal{A}) = \limsup_{R\to\infty} \frac{\mathbb{H}_\varepsilon\left(\mathcal{A}, W_b^{2-\delta,p}(\Omega \cap B_0^R)\right)}{\mathrm{vol}_{\Omega,0}(R)} \tag{6.16}$$

Corollary 6.5. The following estimate is valid:

$$\overline{\mathbb{H}}_\varepsilon(\mathcal{A}) \le C \ln\frac{1}{\varepsilon} \tag{6.17}$$

Indeed, the estimate (6.17) is an immediate corollary of the estimate (6.1) and trivial assertion

$$\lim_{R\to\infty} \frac{\mathrm{vol}_{\Omega,x_0}(R + C_1)}{\mathrm{vol}_{\Omega,x_0}(R)} = 1 \tag{6.18}$$

Definition 6.2. *Let $h_{sp}(\mathcal{A})$ be the following number*

$$h_{sp}(\mathcal{A}) = \limsup_{\varepsilon\to 0} \frac{\overline{\mathbb{H}}_\varepsilon(\mathcal{A})}{\ln\frac{1}{\varepsilon}} \tag{6.19}$$

Corollary 6.6. Let the assumptions of Theorem 6.1 hold. Then

$$h_{sp}(\mathcal{A}) < \infty \tag{6.20}$$

Remark 6.1. The number $h_{sp}(\mathcal{A})$ can be interpreted as some quantitative characteristic of the phenomena of *space* chaotisity of the dynamical system, generated by the equation (0.1). In order to understand this relationship it is worth to compare the definition of h_{sp} with the definition of the topological (time) entropy h_{top} of the dynamical system (see [12]). For the reader convenience we recall shortly this definition. Let M be compact metric space and let $S_t : M \to M$ be a dynamical system (semigroup) on it. For a given $T > 0$ we consider the set $M(0,T) \subset L^\infty([0,T], M)$ of all trajectories $u(t) = S_t u_0$, $t \in [0,T]$ with $u_0 \in M$. Then by definition

$$h_{top} = \limsup_{\varepsilon\to 0} \limsup_{T\to\infty} \frac{\mathbb{H}_\varepsilon\left(M(0,T), L^\infty([0,T], M)\right)}{T} \tag{6.21}$$

§7 Unstable manifolds and lower bounds of ε-entropy.

In previous Section the upper bounds of Kolmogorov's entropy has been obtained. In this Section, using the technique of infinite dimensional unstable manifolds, developed in [9], [24] we obtain the lower bounds of ε-entropy for rather wide class of equations.

We assume in this Section that $\Omega = \mathbb{R}^n$, $g \equiv 0$ and the nonlinearity satisfies the following additional assumptions

$$\begin{cases} 1.\ f \in C^2 \\ 2.\ |f(u, \nabla_x u)| + |f''_{u,u}(u, \nabla_x u)| \le Q(|u|)(1 + |\nabla_x u|^2) \\ 3.\ |f''_{u,\nabla_x u}(u, \nabla_x u)| \le Q(|u|)(1 + |\nabla_x u|^2) \\ 4.\ |f''_{\nabla_x u,\nabla_x u}(u, \nabla_x u)| \le Q(|u|) \end{cases} \tag{7.1}$$

Note, that under the assumptions (0.2) the equation

$$f(z, 0) = \lambda_0 z \tag{7.2}$$

has at least one solution $z_0 = (z_0^1, \cdots, z_0^k) \in \mathbb{R}^k$, consequently, the problem (0.1) always has at least one spatially homogeneous equilibria point.

The main result of this Section is the following Theorem.

Theorem 7.1. *Let the assumptions of Section 2 hold and the space homogeneous equilibria z_0 of the problem (0.1) is exponentially unstable. Then the attractor $\mathcal{A}$ of this problem possesses the following entropy estimates:*

$$C_2R^n \ln\frac{1}{\varepsilon} \le \mathbb{H}_\varepsilon\left(\mathcal{A}, C(B_0^R)\right) \le C_1(R + K\ln\frac{1}{\varepsilon})^n \ln\frac{1}{\varepsilon} \tag{7.3}$$

Moreover, for every $\delta > 0$ there exists $C_\delta > 0$ such that

$$C_\delta\left(\ln\frac{1}{\varepsilon}\right)^{n+1-\delta} \le \mathbb{H}_\varepsilon\left(\mathcal{A}, C(B_0^1)\right) \le C\left(\ln\frac{1}{\varepsilon}\right)^{n+1} \tag{7.4}$$

Proof. The proof of this theorem is based on the estimates (5.13) and (5.14) and on the construction of the infinite dimensional unstable manifold, given in [9], [24] and divided on a number of auxiliary statements formulated below.

Without loss of generality we may assume that $z_0 = 0$. As usual, we consider for the first the linearization of the equation (0.1) near the equilibria point $z_0 = 0$ and study the corresponding nonhomogeneous problem.

$$\partial_t v - Lv = h(t) \tag{7.5}$$

where $Lv \equiv \Delta_x v + Av + B\nabla_x v$, $A = -\lambda_0 - f'_u(0,0)$, $B = -f'_{\nabla_x u}(0,0)$.

Definition 7.1. *Let $\beta > 0$. Then we define the space $\mathbb{L}_\beta$ by the following expression*

$$\mathbb{L}_\beta(E) = \{u \in L^\infty(\mathbb{R}, E) : \|u\|_{\mathbb{L}_\beta} = \sup_{t\le 0} e^{-\beta t}\|u(t), \mathbb{R}^n\|_E < \infty\} \tag{7.6}$$

where E means either Φ_b or $L^p_b(\mathbb{R}^n)$

Proposition 7.1. *Let $\beta > \sigma(L)$ and $h \in \mathbb{L}_\beta(L^p_b)$. Then the equation (7.5) possesses the unique solution $v \in \mathbb{L}_\beta(\Phi_b)$ and consequently defines the linear operator $\mathbb{T}_\beta : \mathbb{L}_\beta(L^p_b) \to \mathbb{L}_\beta(\Phi_b)$, $v(t) = (\mathbb{T}_\beta h)(t)$.*

The proof of Proposition can be derived in a standard way (see [9], for instance). Let us study now the linear homogeneous problem (7.5) ($h \equiv 0$). Applying Fourier transform to the both sides of (7.5) we will have

$$\partial_t \widehat{v}(t) - \widehat{L}(\xi)\widehat{v}(t) = 0 \tag{7.7}$$

where $\widehat{L}(\xi) = -|\xi|^2 + A - iB\xi$. Note that the exponential unstability of the equilibria point $z_0 = 0$ implies that there exists $\xi_0 \in \mathbb{R}^k$ and $\widehat{\lambda}_0 > 0$ such that $\widehat{\lambda}_0 \in \sigma(\widehat{L}(\xi_0))$. Moreover, without loss of generality we may assume that $\sigma(\widehat{L}(\xi)) < \widehat{\lambda}_0 + \varepsilon$ for $\varepsilon > 0$ is small enough and $\xi \in \mathbb{R}^k$.

It follows now from the continuity arguments that there exists a neighbourhood $B^{r_0}_{\xi_0}$ of ξ_0 and functions $\widehat{\lambda}_0 : B^{r_0}_{\xi_0} \to \mathbb{R}$ and $e_0 : B^{r_0}_{\xi_0} \to \mathbb{R}^k$ such that

$$\widehat{L}(\xi)e_0(\xi) = \widehat{\lambda}_0(\xi)e_0(\xi), \quad \xi \in B^{r_0}_{\xi_0} \tag{7.8}$$

and $\widehat{\lambda}_0(\xi) \ge \widehat{\lambda}_0 - \varepsilon$. Moreover, without loss of generality we may assume also that

$$e_0^1(\xi) \equiv 1 \tag{7.9}$$

Let us consider now the class $\mathbb{B}_{\sigma,\xi_0}$ with σ is so small that $\operatorname{supp}\widehat{\phi} \subset B^{r_0}_{\xi_0}$ for every $\phi \in \mathbb{B}_{\sigma,\xi}$. For every $\phi_0 \in \mathbb{B}_{\sigma,\xi_0}$ we define a solution $\widehat{v}(t)$, $t \le 0$ by formula

$$\widehat{v}(t) = e^{\widehat{\lambda}_0(\xi)t}\widehat{\phi}_0(\xi)e_0(\xi) \tag{7.10}$$

In a standard way we deduce from (7.10) that $v \in \mathbb{L}_{\widehat{\lambda}_0 - 2\varepsilon}(\Phi_b)$ and

$$\|v\|_{C_{\widehat{\lambda}_0 - 2\varepsilon}(\Phi_b)} \le C\|\phi_0\|_{\mathbb{B}_{\sigma,\xi_0}} \tag{7.11}$$

Moreover, (7.9) implies that $\Pi_1 v(0) = \phi_0$, where Π_1 is a projection from $\mathbb{R}^k$ to the first component. Thus, we have proved the following proposition.

Proposition 7.2. *Under the assumptions of the theorem there exist $\gamma > 0$, $\xi_0 \in \mathbb{R}^k$, $\sigma > 0$ and a linear operator $P_\gamma : \mathbb{B}_{\sigma,\xi_0} \to \mathbb{L}_\gamma(\Phi_b)$ such that:*
1. $P_\gamma(u_0)(t)$, $t \le 0$ is a solution of the equation (7.5) with $h \equiv 0$;
2. $2\gamma > \sigma(L)$;
3. $\Pi_1 P_\gamma(u_0)(0) = u_0$.
We denote $S_\gamma(u_0) \equiv P_\gamma(u_0)(0)$. Then $\Pi_1 S_\gamma(u_0) = u_0$.

Indeed, the operator T_γ can be defined by (7.10).

Now we are in position to study the neighborhood of zero equilibria point for the nonlinear equation.

Proposition 7.3. *Let the above assumptions be valid and let δ is small enough that $p > \frac{n}{2(1-\delta)}$. Then there exists $\mu_0 > 0$ and a C^1-map*

$$\mathcal{U}_0 \; : \; B(0, \mu_0, \mathbb{B}_{\sigma,\xi_0}) \to \mathcal{A} \tag{7.12}$$

Moreover,

$$\|\mathcal{U}_0(u_0) - S_\gamma(u_0)\|_{W_b^{2-\delta,p}} \le C\|u_0\|^2_{\mathbb{B}_{\sigma,\xi_0}} \tag{7.13}$$

for every $u_0 \in B(0, \mu_0, \mathbb{B}_{\sigma,\xi_0})$

Proof. The proof of this Proposition 7.3 is based on the implicit function theorem and on the following lemma.

Lemma 7.1. *Let f satisfies conditions (7.1) and $f(0,0) = f'_u(0,0) = f'_{\nabla_x u}(0,0) = 0$ Then for every $\beta > 0$ the Nemitskij operator $Fu = f(u, \nabla_x u)$ belongs to the space $C^1(\mathbb{L}_\gamma(W_b^{2-\delta,p}), \mathbb{L}_{2\gamma}(L^p_b))$ if $p > \frac{n}{2(1-\delta)}$.*

The assertion of this lemma can be verified in a direct way (see [24], for example).

Now we are going to find the backward solutions of the problem (0.1) near $z_0 = 0$ equilibria point using the implicit function theorem. To this end we rewrite this equation in the form

$$\partial_t u - Lu = f_{A,B}(u, \nabla_x u), \quad t \le 0 \tag{7.14}$$

where $f_{A,B}(u, \nabla_x u) = f(u, \nabla_x u) - f'_u(0,0)u - f'_{\nabla_x u}\nabla_x u$. Then $f_{A,B}$ satisfies all assumptions of Lemma 7.1 Let us fix γ such as in P=AAoposition 7.2 and consider the equation

$$u + \mathbb{T}_{2\gamma}F_{A,B}u = \mathbb{P}_\gamma u_0, \quad u \in \mathbb{L}_\gamma(\Phi_b) \tag{7.15}$$

where $u_0 \in \mathbb{B}_{\sigma,\xi_0}$ and σ satisfies the conditions of Proposition 7.2. Note that every solution of (7.15) is simultaneously a solution of the equation (7.14) hence it is sufficient to solve (7.15) in $\mathbb{L}_\gamma(\Phi_b)$.

To this end we introduce a function $\mathcal{F} : \mathbb{L}_\gamma(\Phi_b) \times \mathbb{B}_{\sigma,\xi_0} \to \mathbb{L}_\gamma(\Phi_b)$ by formula

$$\mathcal{F}(u, u_0) = u + \mathbb{T}_{2\gamma} F_{A,B} u - \mathbb{P}_\gamma u_0 \tag{7.16}$$

It follows from Propositions 7.1, 7.2 and from Lemma 7.1 that $\mathcal{F} \in C^1(\mathbb{L}_\beta(\Phi_b) \times \mathbb{B}_{\sigma,\xi_0}, \mathbb{L}_\beta(\Phi_b))$ and $D_u\mathcal{F}(0,0) = Id$. Hence due to the implicit function theorem (see [22] for instance) there exists a neighborhood $B(0, \mu_0, \mathbb{B}_{\sigma,\xi_0})$ and a C^1-function

$$\mathcal{U} \,:\, B(0, \mu_0, \mathbb{B}_{\sigma,\xi_0}) \to \mathbb{L}_\gamma(\Phi_b) \tag{7.17}$$

such that $\mathcal{F}(\mathcal{U}(u_0), u_0) \equiv 0$ and consequently $\mathcal{U}(u_0)(t)$ is a backward solution of the problem (0.1). The equation (7.16), Propositions 7.1, 7.2, and Lemma 7.1 imply now that

$$\begin{aligned} \|\mathcal{U}(u_0) - \mathcal{P}_\gamma u_0\|_{\mathbb{L}_{2\gamma}(\Phi_b)} &\le C\|F_{A,B}(\mathcal{U}(u_0))\|_{\mathbb{L}_{2\gamma}(L^p_b)} \le \\ &\le C_1\|\mathcal{U}(u_0)\|^2_{\mathbb{L}_\gamma(\Phi_b)} \le C_2\|u_0\|^2_{\mathbb{B}_{\sigma,\xi_0}} \end{aligned} \tag{7.18}$$

Let us define now $\mathcal{U}_0(u_0) = \mathcal{U}(u_0)\big|_{t=0}$. Then (7.18) together with the evident assertion $(\mathbb{P}_\beta u_0)(0) = S_\gamma u_0$ imply the estimate (7.13). The assertion $\mathcal{U}_0(B(0, \mu_0, \mathbb{B}_{\sigma,\xi_0})) \subset \mathcal{A}$ follows immediately from description (4.2) of the attractor $\mathcal{A}$ and from the fact that the solution $u(t) = \mathcal{U}(u_0)(t)$ of the problem (0.1) which is defined for the first only for $t < 0$ can be extended due to Theorem 2.4 to a complete solution $u(t)$, $t \in \mathbb{R}$ and $u(0) = \mathcal{U}_0(u_0)$. Proposition 7.3 is proved.

Corollary 7.1. *Let $u_0^1, u_0^2 \in B(0, \mu, \mathbb{B}_{\sigma,\xi_0})$ and $\mu \le \mu_0$. Then for every $R > 0$*

$$\|\mathcal{U}_0(u_0^1) - \mathcal{U}_0(u_0^2)\|_{W_b^{2-\delta,p}(B_0^R)} \ge \|u_0^1 - u_0^2\|_{L^\infty(B_0^R)} - C\mu^2 \tag{7.19}$$

with C independent of R.

Indeed,

$$\begin{aligned} &\|\mathcal{U}_0(u_0^1) - \mathcal{U}_0(u_0^2)\|_{\Phi_b(B_0^R)} \ge \\ &\quad \ge \|S_\gamma u_0^1 - S_\gamma u_0^2\|_{\Phi_b(B_0^R)} - \|\mathcal{U}_0(u_0^1) - S_\gamma u_0^1\|_{\Phi_b(\mathbb{R}^n)} + \|\mathcal{U}_0(u_0^2) - S_\gamma u_0^2\|_{\Phi_b(\mathbb{R}^n)} \ge \\ &\quad \ge \|S_\gamma u_0^1 - S_\gamma u_0^2\|_{\Phi_b(B_0^R)} - C_1(\|u_0^1\|^2_{\mathbb{B}_{\sigma,\xi_0}} + \|u_0^2\|^2_{\mathbb{B}_{\sigma,\xi_0}}) \ge \|u_0^1 - u_0^2\|_{L^\infty(B_0^R)} - 2C_1\mu^2 \end{aligned}$$

Here we have used the fact that $\Pi_1 S_\gamma u_0 = u_0$.

Now we are in position to complete the proof of Theorem 7.1. Indeed, let $\varepsilon > 0$ be small enough, $\mu = \left(\frac{\varepsilon}{2C}\right)^{1/2} \le \mu_0$ and $u_0^1, u_2^0 \in B(0, \mu, \mathbb{B}_\sigma)$ such that

$$\|u_0^1 - u_0^2\|_{L^\infty(B_0^R)} \ge \varepsilon \tag{7.20}$$

Then it follows from (7.19) that

$$\|\mathcal{U}(u_0^1) - \mathcal{U}(u_0^2)\|_{W_b^{2-\delta,p}(B_0^R)} \ge \varepsilon/2 \tag{7.21}$$

The estimates (7.20),(7.21) together with the assertion (7.12) imply that

$$\begin{aligned} &\mathbb{H}_{\varepsilon/4}\left(\mathcal{A}, W_b^{2-\delta,p}(B_0^R)\right) \ge \\ &\quad \ge \mathbb{H}_\varepsilon\left(B(0, \left(\frac{\varepsilon}{2C}\right)^{1/2}, \mathbb{B}_{\sigma,\xi_0}), C(B_0^R)\right) = \mathbb{H}_{(2C\varepsilon)^{1/2}}\left(B(0, 1, \mathbb{B}_{\sigma,\xi_0}), C(B_0^R)\right) \end{aligned} \tag{7.22}$$

The estimates (7.3) and (7.4) is an immediate corollary of (5.13) and (5.14) (see also Remark 5.1). Theorem 7.1 is proved.

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