HOMOCLINIC BIFURCATIONS AND DIMENSION OF ATTRACTORS FOR DAMPED NONLINEAR HYPERBOLIC EQUATIONS

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Abstract. A new method of obtaining lower bounds for the attractor's dimension is suggested which involves analysis of homoclinic bifurcations. The method is applied for obtaining sharp estimates of the attractor's dimension for a class of abstract damped wave equations which are beyond the reach of the classical methods.

Introduction

It is well known that the long-time behavior of solutions of partial differential equations arising in mathematical physics can, in many cases, be described in terms of global attractors of the associated semigroups, see [BaV89,ChV02,Hal87,Tem97] and references therein. Moreover, it is also known that for a large class of equations of mathematical physics, including reaction-diffusion equations, Ginzburg-Landau equations, 2D Navier-Stokes system, damped wave equations, etc., the corresponding attractor has finite Hausdorff and fractal dimensions. Thus, although the phase space for such problems is infinite-dimensional, the dynamics on the attractor occurs to be finite-dimensional, hence it can possibly be understood by methods of the qualitative theory of ordinary differential equations. One of crucial questions here is, of course, obtaining more or less realistic estimates for the dimension of the attractor.

The best known upper estimates here are usually obtained based on the concept of Lyapunov dimension $\dim_L(\mathcal{A})$ of the attractor $\mathcal{A}$, see [CF85,Tem97], and on the

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following estimate:

$$\dim_H(\mathcal{A}) \le \dim_F(\mathcal{A}) \le \dim_L(\mathcal{A}), \tag{0.1}$$

where $\dim_H$ and $\dim_F$ denotes the Hausdorff and the fractal dimension respectively, see [CF85,Han96,Tem97,BlI99,ChI02]. The main point here is that the Lyapunov dimension, by its definition, can be explicitly estimated using sufficiently simple volume-contraction arguments, see [Tem97] for details.

Lower bounds for the attractor's dimension are usually based on the observation that the unstable manifold of any equilibrium of the system is always contained in the global attractor $\mathcal{A}$. Consequently, the following estimate is valid:

$$\dim_F(\mathcal{A}) \ge \dim_H(\mathcal{A}) \ge \max_{u_0 \in \mathcal{R}} N^+(u_0), \tag{0.2}$$

where $\mathcal{R}$ is the set of the equilibria of the system, and $N^+(u_0)$ is the instability index of the equilibrium u_0, see e.g.[BaV89] and [Hal87].

We note that this method of obtaining the lower bounds for the attractor's dimension is perfect for the class of gradient systems (or, which is slightly more general, for systems possessing a global Lyapunov function). Indeed, the dynamics in the gradient case is, in a sense, trivial and the dimension of the attractor is determined by the instability indices of the equilibria only, no matter what is the Lyapunov dimension of the attractor and what are the volume-contraction properties. Namely, in this case we have the equality in the second part of (0.2):

$$\dim_H(\mathcal{A}) = \max_{u_0 \in \mathcal{R}} N^+(u_0), \tag{0.3}$$

see e.g. [BaV89] and [Sel89].

There exists, however, a number of important equations of mathematical physics (such as 2D Navier-Stokes system, Ginzburg-Landau equations, non-gradient systems of damped wave equations, etc.), for which the given methods of estimating the attractor's dimension from above and below yield *different* asymptotics for the dimension in terms of physical parameters of the system, see [Tem97, ChV02] and references therein. Which asymptotics is then correct is a long-standing open problem in the theory of attractors. It is also worth to note that all systems mentioned above are far from being gradient and they usually demonstrate a very complicated (e.g. chaotic) dynamical behavior.

In this paper, we present a new method of obtaining lower bounds for the attractor's dimension which exploits explicitly the *recurrent* (as opposed to a gradient-like) nature of the system, and which is based on some general ideas from the theory of homoclinic bifurcations. Namely, we suggest to estimate from below the attractor's dimension in terms of the maximum $M(\Gamma, u_0)$ of the dimension of the unstable manifold over the periodic orbits which can be born at a bifurcation of a homoclinic orbit Γ to an equilibrium u_0:

$$\dim_F(\mathcal{A}) \ge \dim_H(\mathcal{A}) \ge M(\Gamma, u_0). \tag{0.4}$$

To be more precise, one should consider a family of systems which depend on some set of parameters μ; then the global attractor is a function of μ as well, and (0.4) should be interpreted as

$$\limsup_{\mu \to \mu_0} \dim_F(\mathcal{A}_\mu) \ge \limsup_{\mu \to \mu_0} \dim_H(\mathcal{A}_\mu) \ge M(\Gamma, u_0)$$

where the bifurcational moment $\mu = \mu_0$ corresponds to the existence of the homoclinic loop Γ. Of course, one may use various homo/heteroclinic cycles with the same purposes – we take a homoclinic loop as a simplest possible construction.

As it is argued in [Tur96], for many cases of homoclinic bifurcations the dimension $M(\Gamma, u_0)$ essentially coincides with the Lyapunov dimension of the corresponding equilibrium u_0:

$$M(\Gamma, u_0) \sim \dim_L(u_0), \tag{0.5}$$

no matter how small the dimension $N^+(u_0)$ of the unstable manifold of u_0 is. Thus, under this approach, both upper and lower bounds for the attractor's dimension are given in terms of Lyapunov dimension. That is why we expect this method to be effective in order to obtain sharp bounds for the dimension. Of course, the existence of a homoclinic orbit and the possibility to perturb it in the desired way within the class of systems under consideration is crucial for this method. However, the homoclinic phenomena are so typical for dynamical systems with a non-trivial behavior, that it would be natural to expect that in a wide class of equations of mathematical physics which demonstrate chaotic behavior appropriate homoclinic bifurcations can indeed be detected.

We illustrate our method by a model example of an abstract damped wave equation

$$\partial_t^2 u + \gamma \partial_t u + \mathrm{A}\, u = F(u, \partial_t u) \tag{0.6}$$

in a Hilbert space H. We assume that $\mathrm{A} : D(\mathrm{A}) \to \mathrm{H}$ is a positive self-adjoint operator in H with compact inverse, whose eigenvalues satisfy the estimate

$$C_1 i^{2\kappa} \le \lambda_i \le C_2 i^{2\kappa}, \quad i \in \mathbb{N}, \tag{0.7}$$

for some positive C_1, C_2 and κ. Natural examples for A are provided by elliptic differential operators in a bounded domain, with $\mathrm{H} = L^2$. The quantity $\gamma > 0$ in (0.7) is a dissipation (or damping) parameter which is assumed to be small. It is also assumed that the nonlinear operator $F = F(u, \partial_t u)$ belongs to some class $\mathbb{S}$ of very regular ("smoothing") operators which will be specified in Section 2.

We prove that under the above assumptions, equation (0.6) possesses a global attractor $\mathcal{A}$ in the corresponding energetic space E, and the Lyapunov dimension of the attractor satisfies

$$C_1' \gamma^{-1} \le \dim_L(\mathcal{A}) \le C_2' \gamma^{-1}, \tag{0.8}$$

for some positive constants $C_{1,2}'$ which are independent of γ. Consequently, due to (0.1), we have

$$\dim_F(\mathcal{A}) \le C_2' \gamma^{-1}. \tag{0.9}$$

On the other hand, when the nonlinearity belongs to the class $\mathbb{S}$ of very regular operators, we show that for every $\varepsilon > 0$ there exists a positive constant C_ε such that

$$\max_{u_0 \in \mathcal{R}} N^+(u_0) \le C_\varepsilon \gamma^{-\varepsilon}.$$

Thus, using the classical methods of estimating the dimension of the attractor $\mathcal{A}$ (which are based on (0.1) and (0.2)), we will necessarily have a tremendous gap between the asymptotics for upper and lower bounds of the attractor's dimension.

Nevertheless, using our "homoclinic" method, we construct nonlinearities F belonging to the same class $\mathbb{S}$, for which we have

$$\dim_F(\mathcal{A}) \geq \dim_H(\mathcal{A}) \geq C_3\gamma^{-1}, \tag{0.10}$$

for some positive constant C_3. Thus, at least in the case of damped hyperbolic equations with smoothing nonlinearities, the correct asymptotics for the dimension of the attractor is given by the corresponding asymptotics of the Lyapunov dimension, and the estimate (0.2) is not very much relevant.

Note also that $\mathcal{A}$ is the so-called maximal attractor, so it could be possible, in principle, that the dimension of $\mathcal{A}$ can be decreased drastically by removing from $\mathcal{A}$ non-recurrent orbits (like in gradient-like systems where such operation reduces $\mathcal{A}$ to a zero-dimensional set, typically). We show, however, that nonlinearities $F \in \mathbb{S}$ exist for which equation (0.6) has a *minimal* set whose dimension satisfies (0.10); therefore, the Lyapunov dimension (up to a constant factor) measures the complexity of the dynamics of damped hyperbolic equations correctly.

The examples which we are talking about are obtained as small perturbations of a decoupled system of second order ODE's (see (4.7)) which is an infinite collection of damped linear oscillators (with the damping of order γ) plus a single one degree of freedom Hamiltonian system describing a particle in a double-well potential on a straight line. It may be counter-intuitive, because none of the modes here shows a chaotic behavior and, moreover, all of them but one are damped, but we show that for any fixed $\gamma > 0$, an interaction of an arbitrarily small (in comparison with γ) strength can be arranged between these modes such that an extremely complicated behavior is ignited, involving a huge ($\sim 1/\gamma$) number of modes (see Remark 4.3). Note that we nowhere use the linear character of the oscillatory modes and our construction works for a chain of nonlinear damped oscillators as well. Therefore, our results should be applicable to perturbations of other integrable equations, such as the nonlinear Shrödinger equation, etc..

The paper is organized as follows. The existence of the global attractor for problem (0.6) is verified in Section 1. The upper bounds for fractal and Lyapunov dimension of this attractor are obtained in Section 2. The quantity $M(\Gamma, u_0)$ (the maximal possible number of nonnegative Lyapunov exponents for periodic orbits which can be born at a bifurcation of the homoclinic loop Γ) is computed for a special class of homoclinic loops in Section 3. Finally, in Section 4, we show that such homoclinic orbits really appear in equations (0.6) with the nonlinearities belonging to the class $\mathbb{S}$, then based on (0.4) we derive sharp lower bounds for the attractor's dimension.

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§1 Abstract nonlinear hyperbolic equation and its attractor.

In this Section, we study the following abstract nonlinear hyperbolic equation in a Hilbert space H:

$$
\begin{cases} \partial_t^2 u + \gamma \partial_t u + \mathrm{A}\, u = F(u, \partial_t u), \\ u\big|_{t=0} = u_0, \quad \partial_t u\big|_{t=0} = u_0', \end{cases} \tag{1.1}
$$

where $u = u(t)$ is an unknown H-valued function, $\mathrm{A} : D(\mathrm{A}) \to \mathrm{H}$ is a given positive selfadjoint operator in H with compact inverse, $\gamma > 0$ is a given positive number which is assumed to be small and F is a given nonlinear operator.

As usual (see e.g. [GhT87,Tem97]), we define a scale H^s of Hilbert spaces associated with H via

$$
\mathrm{H}^s := D((\mathrm{A})^{s/2}), \quad \|u\|^2_{\mathrm{H}^s} := \|(\mathrm{A})^{s/2} u\|^2_{\mathrm{H}} = ((\mathrm{A})^s u, u) \tag{1.2}
$$

(here and below $(\cdot,\cdot)$ denotes the inner product in H) and consider equation (1.1) as an evolution equation with respect to $\xi(t) = [u(t), \partial_t u(t)]$ in the corresponding energetic phase spaces

$$
\mathrm{E}^s := \mathrm{H}^{s+1} \times \mathrm{H}^s, \quad \xi(t) := [u(t), \partial_t u(t)] \in \mathrm{E}^s \tag{1.3}
$$

(in fact, we will consider only the case where the initial data $[u_0, u_0']$ belong either to the space $\mathrm{E} := \mathrm{E}^0$, or to E^1).

It is also assumed that the nonlinear term F belongs to the space

$$
F \in C^1_b(\mathrm{E}, \mathrm{H}) \tag{1.4}
$$

and its partial derivatives F'_u and $F'_{\partial_t u}$ satisfy the following conditions:

$$
\begin{cases} 1. \ \ (F'_v(u,v)\theta, \theta) \le \frac{\gamma}{8}\|\theta\|^2_{\mathrm{H}} + C_\gamma \|\theta\|^2_{\mathrm{H}^{-1}}, \\ 2. \ \ |(F'_u(u,v)w, \theta)| \le \frac{\gamma}{8}\left(\|w\|^2_{\mathrm{H}^1} + \|\theta\|^2_{\mathrm{H}}\right) + C_\gamma \|\theta\|^2_{\mathrm{H}^{-1}}, \end{cases} \tag{1.5}
$$

where $[u,v], [w,\theta] \in \mathrm{E}$ are arbitrary, $\gamma > 0$ is the same as in equation (1.1), and the constant C_γ is independent of $[u,v]$ and $[w,\theta]$.

The following theorem shows that under the above assumptions equation (1.1) generates a dissipative semigroup in the energetic space E.

Theorem 1.1. *Let the assumptions* (1.4) *and* (1.5) *hold. Then, for every* $\xi(0) := [u(0), \partial_t u(0)] \in \mathrm{E}$, *equation* (1.1) *has a unique global solution* $\xi(t) \in C([0,\infty), \mathrm{E})$ *and the following estimate is valid:*

$$
\|\xi(t)\|^2_{\mathrm{E}} \le C\|\xi(0)\|^2_{\mathrm{E}} e^{-\gamma t} + C_1, \tag{1.6}
$$

where the constants C *and* C_1 *depend only on* F, A *and* γ. *Consequently, equation* (1.1) *generates a semigroup*

$$
S_t : \mathrm{E} \to \mathrm{E}, \quad \textit{by} \ \ S_t \xi(0) := \xi(t). \tag{1.7}
$$

Moreover, this semigroup is globally Lipschitz continuous with respect to the initial data $[u(0), \partial_t u(0)] \in \mathrm{E}$, *i.e.*

$$
\|\xi_1(t) - \xi_2(t)\|^2_{\mathrm{E}} \le C e^{Kt} \|\xi_1(0) - \xi_2(0)\|^2_{\mathrm{E}}, \tag{1.8}
$$

where K and C depend only on A, γ and F (and they are independent of the solutions $u_1(t)$ and $u_2(t)$ of problem (1.1)).

If $\xi(0)\in \mathrm{E}^1$, then the corresponding solution $\xi(t)$ belongs to E^1 for every $t\ge 0$ and satisfies the following estimate:

$$\|\xi(t)\|^2_{\mathrm{E}^1}\le C_2\|\xi(0)\|^2_{\mathrm{E}^1}e^{-\gamma t/8}+C_3, \tag{1.9}$$

for some positive constants C_2 and C_3 which depend only on A, F and γ.

The proof of this theorem is quite standard, so we move it to Appendix A.

Let us now verify that the semigroup $S_t:\mathrm{E}\to\mathrm{E}$ possesses a global compact attractor in the phase space E. Recall that the set $\mathcal{A}\subset \mathrm{E}$ is called a global attractor for the semigroup $S_t:\mathrm{E}\to\mathrm{E}$ if the following conditions are satisfied:

1. $\mathcal{A}$ is compact in E;
2. $\mathcal{A}$ is strictly invariant with respect to S_t, i.e. $S_t\,\mathrm{E}=\mathrm{E}$;
3. $\mathcal{A}$ attracts bounded subsets of E as $t\to\infty$, i.e. for every bounded $B\subset \mathrm{E}$ and every neighborhood $\mathcal{O}(\mathcal{A})$ in E, there exists a number $T=T(\|B\|_{\mathrm{E}},\mathcal{O})$ such that

$$S_tB\subset\mathcal{A},\quad \text{for } t\ge T \tag{1.10}$$

(see e.g. [BaV89,Tem97] for details).

Theorem 1.2. *Let the assumptions of Theorem 1.1 hold. Then semigroup (1.7) associated with the nonlinear hyperbolic problem (1.1) possesses a global attractor $\mathcal{A}$, which is bounded in E^1. This attractor is generated by all complete bounded solutions of (1.1):*

$$\mathcal{A}=\{\xi(0),\quad \xi(t):=[u(t),\partial_t u(t)],\ t\in\mathbb{R}\ \textit{solves}\ (1.1)\ \textit{and}\ \|\xi(t)\|_{\mathrm{E}}\le C_u,\ t\in\mathbb{R}\}. \tag{1.11}$$

Proof. According to the abstract attractor's existence theorem (see e.g. [BaV89]) the theorem will be proven if we verify the following conditions on the semigroup S_t:

1. $S_t:\mathrm{E}\to\mathrm{E}$ is continuous with respect to $\xi(0)$ for every fixed $t\ge 0$;
2. The semigroup S_t possesses a compact attracting (in the sense of (1.10)) set $K\subset\subset \mathrm{E}$.

Let us verify these conditions. Indeed, the continuity of S_t is given by Theorem 1.1. So, we are left to verify the second condition. To this end, we split the solution $u(t)$ of (1.1) as follows: $u(t):=v(t)+w(t)$, where $v(t)$ is a solution of the following problem:

$$\begin{cases}\partial_t^2 v+(\gamma+b\,\mathrm{A}^{-1})\partial_t v+\mathrm{A}\,v+Mv-F(v,\partial_t v)=Mu(t)+b\,\mathrm{A}^{-1}\,\partial_t u(t),\\ [v,\partial_t v]\big|_{t=0}=0.\end{cases} \tag{1.12}$$

Here $M\gg 1$ and $b\gg 1$ are sufficiently large positive constants which will be specified below.

Consequently, the rest function $w(t)$ satisfies the equation

$$\begin{cases}\partial_t^2 w+(\gamma+b\,\mathrm{A}^{-1})\partial_t w+\mathrm{A}\,w+Mw=l^1(t)w+l^2(t)\partial_t w,\\ [w,\partial_t w]\big|_{t=0}=\xi(0),\end{cases} \tag{1.13}$$

where

$$
(1.14)\qquad \begin{aligned}
l^1(t) &:= \int_0^1 F'_u(su(t)+(1-s)v(t), s\partial_t u(t)+(1-s)\partial_t v(t))\,ds,\\
l^2(t) &:= \int_0^1 F'_{\partial_t u}(su(t)+(1-s)v(t), s\partial_t u(t)+(1-s)\partial_t v(t))\,ds.
\end{aligned}
$$

We will prove that $\|w(t)\|_E$ tends uniformly (with respect to small variations in initial conditions) to zero, and $v(t)$ enters some fixed ball in E^1 as time grows. This ball is compact in E, so it can be taken as a desired attracting set K.

Let us first estimate $w(t)$.

Lemma 1.1. *Let the assumptions of Theorem 1.1 hold. Then, there exist large positive constants $M = M(\gamma, F, \mathrm{A})$ and $b = b(\gamma, F, \mathrm{A})$ such that the solution $w(t)$ of equation* (1.13) *satisfies the following estimate:*

$$
(1.15)\qquad \|[w(t), \partial_t w(t)]\|_{\mathrm{E}} \le C' e^{-\alpha t}\|\xi(0)\|_{\mathrm{E}},
$$

for appropriate positive constants C' and α which are independent of u.

Proof. Taking the inner product in H of equation (1.13) with $\partial_t w + \frac{\gamma + b\,\mathrm{A}^{-1}}{2} w(t)$, we derive the following relation:

$$
(1.16)\quad \begin{aligned}
&\partial_t\big\{\|\partial_t w\|^2_{\mathrm{H}} + \|w\|^2_{\mathrm{H}^1} + \big((\gamma + b\,\mathrm{A}^{-1})w, \partial_t w\big) + M\|w\|^2_{\mathrm{H}}\big\} +\\
&+ \frac{\gamma}{2}\big\{\|\partial_t w\|^2_{\mathrm{H}} + \|w\|^2_{\mathrm{H}^1} + M\|w\|^2_{\mathrm{H}} + \big((\gamma + b\,\mathrm{A}^{-1})w, \partial_t w\big)\big\} = -b\|\partial_t w\|^2_{\mathrm{H}^{-1}} -\\
&- b\|w\|^2_{\mathrm{H}} - Mb\|w\|^2_{\mathrm{H}^{-1}} - \frac{\gamma}{2}\left(\|\partial_t w\|^2_{\mathrm{H}} + \|w\|^2_{\mathrm{H}^1} + M\|w\|^2_{\mathrm{H}}\right) +\\
&+ 2\left(l^1(t)w, \partial_t w\right) + 2\left(l^2(t)\partial_t w, \partial_t w\right) +\\
&+ \left(l^1(t)w, (\gamma + b\,\mathrm{A}^{-1})w\right) + \left(l^2(t)\partial_t w, (\gamma + b\,\mathrm{A}^{-1})w\right) -\\
&- \frac{1}{2}\left((\gamma + b\,\mathrm{A}^{-1})\partial_t w, (\gamma + 2b\,\mathrm{A}^{-1})w\right) \equiv h_w(t).
\end{aligned}
$$

We recall that, by conditions (1.4) and formulas (1.14),

$$
(1.17)\qquad \|l^2(t)\|_{\mathcal{L}(\mathrm{H},\mathrm{H})} + \|l^1(t)\|_{\mathcal{L}(\mathrm{H}^1,\mathrm{H})} \le C,
$$

where C is independent of u and t, and, by conditions (1.5),

$$
(1.18)\qquad \left(l^2(t)\partial_t w(t), \partial_t w(t)\right) \le \frac{\gamma}{8}\|\partial_t w\|^2_{\mathrm{H}} + C_\gamma\|\partial_t w(t)\|^2_{\mathrm{H}^{-1}}.
$$

Estimating the right-hand side $h_w(t)$ of (1.16) by Hölder inequality and taking into account estimates (1.17) and (1.18), we obtain

$$
(1.19)\qquad h_w(t) \le \left(C_\gamma - \frac{1}{2}b\right)\|\partial_t w(t)\|^2_{\mathrm{H}^{-1}} + \left(C'_\gamma(1+b^3) - M\right)\|w\|^2_{\mathrm{H}},
$$

where C_γ and C'_γ are two positive constants which depend only on γ, F and A, but are independent of b, M and u. Fixing now the constants M and b in such a way that

$$
b = 2C_\gamma, \quad M \ge C'_\gamma(1+b^3),
$$

we obtain the inequality $h_w(t)\le 0$. Moreover, without loss of generality we assume that M is chosen in such a way that, in addition,

$$|\left((\gamma+b\,\mathrm{A}^{-1})w,\partial_t w\right)|\le\frac{1}{2}\left(\|\partial_t w\|^2_{\mathrm{H}}+M\|w\|^2_{\mathrm{H}}\right).$$

Applying now the Gronwall's inequality to (1.16), we obtain (1.15). Lemma 1.1 is proven.

Now we are ready to estimate the solution $v(t)$ of equation (1.12). We rewrite this equation in the following equivalent form:

$$\begin{cases}\partial_t^2 v+\gamma\partial_t v+\mathrm{A}\,v-F(v,\partial_t v)=Mw(t)+b\,\mathrm{A}^{-1}\,\partial_t w(t):=h_{M,b}(t),\\ [v,\partial_t v]\big|_{t=0}:=0,\end{cases}\tag{1.20}$$

We note that equation (1.20) is a nonautonomous analogue of equation (1.1). Moreover, due to Theorem 1.1 and Lemma 1.1, the function $h_{M,b}(t)$ can be estimated as follows:

$$\|h_{M,b}(t)\|^2_{\mathrm{H}^1}+\|\partial_t h_{M,b}(t)\|^2_{\mathrm{H}}\le C_{M,b}\left(1+\|\xi(0)\|^2_{\mathrm{E}}e^{-\alpha t}\right),\tag{1.21}$$

for an appropriate constant $C_{M,b}$ which is independent of u. Consequently, using estimate (1.21) and the fact that $v(0)=0$, $\partial_t v(0)=0$, arguing exactly as in the proof of Theorem 1.1 (see Appendix A), we obtain that the solution $[v,\partial_t v]$ of (1.12) belongs to the space $C(\mathbb{R}_+,\mathrm{E}^1)$ and satisfies the estimate

$$\|[v(t),\partial_t v(t)]\|_{\mathrm{E}^1}\le C_*\left(\|\xi(0)\|_{\mathrm{E}}e^{-\alpha t}+1\right),\tag{1.22}$$

for some positive constants α and C_* which depend on M and γ, but are independent of u.

Estimates (1.15) and (1.22) imply that the set

$$K:=\{\xi\in\mathrm{E}^1,\ \|\xi\|_{\mathrm{E}^1}\le 2C^*\}$$

is a compact (in E) attracting set for the semigroup S_t. Thus, all conditions of the abstract attractor's existence theorem are verified and Theorem 1.2 is proven.

Remark 1.1. We recall that our conditions (1.4) and (1.5) imply that the operator F (along with its first derivatives F'_u and $F'_{\partial_t u}$) is globally bounded as $\|\xi\|_{\mathrm{E}}\to\infty$. This is enough for our purposes since in our examples of sharp upper and lower bounds for the attractor's dimension (see Section 4) the nonlinearity F has a bounded support. So, for simplicity, we restrict ourselves to the class of globally bounded nonlinearities, although more general nonlinearities (see e.g. [GhT87,Fei95,Tem97]) can be treated in the same way.

Remark 1.2. We note that conditions (1.4) and (1.5) are, obviously, satisfied if

$$F\in C^1_b(\mathrm{E}^{-\delta},\mathrm{H}),\tag{1.23}$$

for some *positive* exponent δ. In the sequel, we will often use this more strong condition (1.23) instead of conditions (1.4) and (1.5).

§2 Upper bounds for the attractor's dimension.

In this Section we show, using the standard volume contraction technique, that the attractor $\mathcal{A}$ of (1.1) constructed in the previous Section has finite Hausdorff and fractal dimensions, and we obtain some estimates for this dimension in terms of the dissipative parameter γ. To this end, we need the following assumption: there is an exponent $\kappa > 0$ and two positive constants C_1 and C_2 such that

$$C_1 i^{2\kappa} \le \lambda_i \le C_2 i^{2\kappa}, \quad i \in \mathbb{N}, \tag{2.1}$$

where $0 < \lambda_1 \le \lambda_2 \le \cdots$ are the eigenvalues of the operator A.

Remark 2.1. We note that assumption (2.1) is always satisfied if A is an elliptic differential operator in a bounded domain $\Omega \subset \mathbb{R}^n$ with a sufficiently smooth boundary and $\mathrm{H} := L^2(\Omega)$ (see e.g. [Tri78]). Moreover, in this case $\kappa := \frac{k}{2n}$, where k is the order of A.

In order to formulate the abstract theorem for estimating the dimension of invariant sets, we need the following definition.

Definition 2.1. A map $S : \mathcal{A} \to \mathcal{A}$, where $\mathcal{A}$ is a subset of a certain Banach space E is called uniformly quasidifferentiable on $\mathcal{A}$ if for any $\xi \in \mathrm{E}$ there is a linear operator $S'(\xi) : \mathrm{E} \to \mathrm{E}$ (the quasidifferential) such that

$$\|S(\xi_1) - S(\xi_2) - S'(\xi_1)(\xi_1 - \xi_2)\|_{\mathrm{E}} = o(\|\xi_1 - \xi_2\|_{\mathrm{E}}) \tag{2.2}$$

holds uniformly with respect to $\xi_1, \xi_2 \in \mathcal{A}$. It is also assumed that

$$\sup_{\xi\in\mathcal{A}} \|S'(\xi)\|_{\mathcal{L}(\mathrm{E},\mathrm{E})} < \infty \quad \text{and} \quad S' \in C(\mathcal{A}, \mathcal{L}(\mathrm{E},\mathrm{E})). \tag{2.3}$$

Theorem 2.1. *Let S_t be a semigroup in a certain Hilbert space* E *and let $\mathcal{A} \subset \mathrm{E}$ be a compact strictly invariant set of this semigroup ($S_t\mathcal{A} = \mathcal{A}$). Let us suppose also that the semigroup S_t is uniformly quasidifferentiable on $\mathcal{A}$ for some fixed $t = T$ and the following inequality holds for some positive integer d:*

$$\omega_d(\mathcal{A}) := \sup_{\xi\in\mathcal{A}} \omega_d(S'_T(\xi)) < 1, \tag{2.4}$$

where $\omega_d(L) := \|\Lambda^d L\|_{\Lambda^d \mathrm{E}}$ is the norm of the d-th exterior power of the operator L in the Hilbert space $\Lambda^d \mathrm{E}$ (see e.g. [Tem97]*). Then the fractal dimension of $\mathcal{A}$ is finite in* E*. Moreover,*

$$\dim_H(\mathcal{A}, \mathrm{E}) \le \dim_F(\mathcal{A}, \mathrm{E}) \le d. \tag{2.5}$$

For the proof of this theorem see [Tem97] for the case of Hausdorff dimension and [Han96,BlI99,ChI02] for the fractal dimension.

Lemma 2.1. *Let the assumptions of Theorem 1.1 hold. Then, semigroup* (1.7)*, associated with hyperbolic equation* (1.1) *is uniformly quasidifferentiable on the attractor $\mathcal{A}$ and its quasidifferential $S'_t(\xi(0))$ at $\xi(0) \in \mathcal{A}$ is defined via the following standard expression:*

$$S'_t(\xi(0))\eta := [v(t), \partial_t v(t)], \quad \text{where } \eta \in \mathrm{E} \tag{2.6}$$

and $v(t)$ is a solution of the equation of variations

$$(2.7)\qquad \begin{cases} \partial_t^2 v + \gamma\partial_t v + \mathrm{A}\, v = F'_u(u(t),\partial_t u(t))v(t) + F'_{\partial_t u}(u(t),\partial_t u(t))\partial_t v(t),\\ [v,\partial_t v]\big|_{t=0} = \eta, \quad [u(t),\partial_t u(t)] := S_t\xi(0).\end{cases}$$

The assertion of the lemma is completely standard, so we move its proof into Appendix A.

Thus, according to Theorem 2.1, for estimating the dimension of $\mathcal{A}$ it is sufficient to estimate the norms of d-external powers for solving operator of equation of variations associated with the hyperbolic problem (1.1). To this end, following [GhT87], we introduce a new variable $\theta(t) := \partial_t u + \frac{\gamma}{2}u(t)$ and rewrite system (1.1) in the equivalent form in variables $[u,\theta] \in \mathrm{E}$. We obtain

$$(2.8)\qquad \partial_t \begin{pmatrix} u \\ \theta \end{pmatrix} = \begin{pmatrix} -\frac{\gamma}{2}u + \theta \\ F(u,\theta - \frac{\gamma}{2}u) + \frac{\gamma^2}{4}u - \mathrm{A}\, u - \frac{\gamma}{2}\theta \end{pmatrix}.$$

Thus, instead of applying Theorem 2.1 to the initial system in $[u,\partial_t u]$ variables we will use it for the transformed system (2.8) (since these systems are linearly equivalent, they have equivalent attractors whose dimensions coincide).

The equation of variations for the transformed system, obviously, has the form

$$(2.9)\qquad \partial_t\eta(t) = \mathbb{L}(u(t),\partial_t u(t))\eta(t), \quad [u(t),\partial_t u(t)] := S_t\xi(0),$$

where

$$(2.10)\qquad \mathbb{L} := \begin{pmatrix} -\frac{\gamma}{2} & ; & 1 \\ -\mathrm{A} + \frac{\gamma^2}{4} + F'_u(\xi) - \frac{\gamma}{2}F'_{\partial_t u}(\xi) & ; & -\frac{\gamma}{2} + F'_{\partial_t u}(\xi) \end{pmatrix}.$$

In order to estimate the exterior powers of solving operator $S'_T(\xi(0)) : \eta \to \eta(T)$ of linear problem (2.9), we use the following standard lemma (see [GhT87] or [Tem97]).

Lemma 2.2. *Let the assumptions of Lemma 2.1 hold. Then*

$$(2.11)\qquad \omega_d(S'_T(\xi(0)) \le e^{\int_0^T \mathrm{Tr}_d\{\mathbb{L}(\xi(t))\}\,dt}, \quad \xi(t) := S_t\xi(0),$$

where $\mathrm{Tr}_d(L)$ means the d-dimensional trace of the operator $L : \mathrm{E} \to \mathrm{E}$ in E, i.e.

$$(2.12)\qquad \mathrm{Tr}_d(L) := \sup\Big\{ \sum_{i=1}^d (L\eta_i,\eta_i)_{\mathrm{E}} : \ \|\eta_i\|_{\mathrm{E}} = 1, \quad (\eta_i,\eta_j)_{\mathrm{E}} = 0 \ \ \textit{for } i \ne j \Big\}.$$

Now we are in a position to estimate the fractal dimension of the attractor $\mathcal{A}$ of hyperbolic equation (1.1). For simplicity, we assume that the nonlinearity F satisfies condition (1.23) and estimate the corresponding dimension in terms of parameters γ, κ (introduced in (2.1)) and δ (introduced in (1.23)). (In the general case of conditions (1.4) and (1.5), this dimension can be analogously estimated in terms of γ, κ and the constant C_γ defined in (1.5).)

Theorem 2.2. *Let the assumptions of Theorem 1.1 hold and let, in addition, the nonlinearity F satisfies* (1.23). *Then, the fractal dimension of the attractor $\mathcal{A}$ is finite in* E *and can be estimated, as $\gamma \to 0$, as follows*

$$\dim_F(\mathcal{A}, \mathrm{E}) \le CN(\gamma) := C \begin{cases} \gamma^{-\frac{2}{\kappa\delta}} & , \quad \kappa\delta < 1, \\ \gamma^{-2} \ln \frac{1}{\gamma} & , \quad \kappa\delta = 1, \\ \gamma^{-2} & , \quad \kappa\delta > 1, \end{cases} \tag{2.13}$$

where C is independent of $\gamma \to 0$, $\delta > 0$ and $\kappa > 0$.

Proof. According to Theorem 2.1 and Lemma 2.2, it is sufficient to prove that

$$\sup_{\xi \in \mathcal{A}} \mathrm{Tr}_d\{\mathbb{L}(\xi)\} < 0, \quad \text{for some } d \le CN(\gamma). \tag{2.14}$$

In order to show this, we first estimate the quadratic form, associated with the operator $\mathbb{L}$, using the assumptions (1.23) and Schwartz inequality

$$\begin{aligned} (\mathbb{L}\eta, \eta)_{\mathrm{E}} = &-\frac{\gamma}{2}\|\eta_u\|^2_{\mathrm{H}^1} + (\eta_\theta, \mathrm{A}\,\eta_u) - (\eta_\theta, \mathrm{A}\,\eta_u) + \frac{\gamma^2}{4}(\eta_u, \eta_\theta) + \\ &+ \Big((F'_u - \frac{\gamma}{2}F'_{\partial_t u})\eta_u, \eta_\theta\Big) - \frac{\gamma}{2}\|\eta_\theta\|^2_{\mathrm{H}} + \big(F'_{\partial_t u}\eta_\theta, \eta_\theta\big) \le \\ &\le -\frac{\gamma}{4}\left(\|\eta_u\|^2_{H^1} + \|\eta_\theta\|^2_{\mathrm{H}}\right) + \tilde{C}\gamma^{-1}\left(\|\eta_u\|^2_{\mathrm{H}^{1-\delta}} + \|\eta_\theta\|^2_{\mathrm{H}^{-\delta}}\right) := (\mathcal{B}\eta, \eta)\,, \end{aligned} \tag{2.15}$$

where $\eta := [\eta_u, \eta_\theta] \in \mathrm{E}$, the operator $\mathcal{B}$ is defined as

$$\mathcal{B} := \frac{\gamma}{4}\begin{pmatrix} -\mathrm{Id} + 4\tilde{C}\gamma^{-2}\,\mathrm{A}^{-\delta/2} & ; & 0 \\ 0 & ; & -\mathrm{Id} + 4\tilde{C}\gamma^{-2}\,\mathrm{A}^{-\delta/2} \end{pmatrix}$$

and the constant $\tilde{C}$ is independent of γ, δ and $\eta \in \mathrm{E}$.

It follows now from (2.15) that for any $d \in \mathbb{N}$,

$$\mathrm{Tr}_d\{\mathbb{L}\} \le \mathrm{Tr}_d\{\mathcal{B}\}. \tag{2.16}$$

We now observe that the operator $\mathcal{B}$ is selfadjoint, hence, by the classical min – max principle (see e.g. [Tem97]), its traces can be immediately expressed in terms of its eigenvalues, namely,

$$\mathrm{Tr}_d\{\mathcal{B}\} = \frac{\gamma}{2}\left(-d + 4\tilde{C}\gamma^{-2}\sum_{i=1}^{d}\lambda_i^{-\delta/2}\right), \tag{2.17}$$

where λ_i are the eigenvalues of A. The estimate (2.14) is an immediate corollary of (2.15)–(2.17) and of the assumption (2.1) on the asymptotics of λ_i. Theorem 2.2 is proven.

The following theorem shows that the estimate (2.13) can be essentially improved if the additional regularity of the nonlinear term F is known.

Theorem 2.3. *Let the assumptions of Theorem 1.1 hold and let, in addition,*

$$F \in C^1_b(\mathrm{E}^{-s-1/2}, \mathrm{H}^{s+1/2}), \tag{2.18}$$

where $s > -1/2$ is some regularity exponent. Then, the dimension of the corresponding attractor $\mathcal{A}$ can be estimated via

$$\dim_F(\mathcal{A}, \mathrm{E}) \le C_1 \begin{cases} \gamma^{-\frac{2}{\kappa(2s+1)}} & , \quad \kappa(s+1/2) < 1, \\ \gamma^{-1} \ln \frac{1}{\gamma} & , \quad \kappa(s+1/2) = 1, \\ \gamma^{-1} & , \quad \kappa(s+1/2) > 1, \end{cases} \tag{2.19}$$

where the constant C_1 depends on s and F, but is independent of γ.

Proof. Indeed, due to condition (2.18), we have the following estimates:

$$\begin{aligned} |\,(F'_u(u,\partial_t u)\eta_u, \eta_\theta)\,| \le \|F'_u(u,\partial_t u)\eta_u\|_{\mathrm{H}^{s+1/2}} \|\eta_\theta\|_{\mathrm{H}^{-s-1/2}} \le \\ \le C\left(\|\eta_u\|^2_{\mathrm{H}^{-s+1/2}} + \|\eta_\theta\|^2_{\mathrm{H}^{-s-1/2}}\right), \end{aligned} \tag{2.20}$$

and, analogously,

$$\left(F'_{\partial_t u}(u,\partial_t u)\eta_\theta, \eta_\theta\right) \le \|F'_{\partial_t u}(u,\partial_t u)\eta_\theta\|_{\mathrm{H}^{s+1/2}} \|\eta_\theta\|_{\mathrm{H}^{-s-1/2}} \le C\|\eta_\theta\|^2_{\mathrm{H}^{-s-1/2}}, \tag{2.21}$$

where the constant C depends only on F. Estimates (2.20) and (2.21) allow to improve (2.15) in the following way:

$$(\mathbb{L}\eta, \eta)_{\mathrm{E}} \le (\mathcal{B}_s \eta, \eta)_{\mathrm{E}},$$

where

$$\mathcal{B}_s := \frac{\gamma}{4}\begin{pmatrix} -\operatorname{Id} + \tilde{C}\gamma^{-1}\mathrm{A}^{-(s+1/2)/2} & ; & 0 \\ 0 & ; & -\operatorname{Id} + \tilde{C}\gamma^{-1}\mathrm{A}^{-(s+1/2)/2} \end{pmatrix},$$

for some constant $\tilde{C} > 0$ which is independent of γ (the term $\gamma\left(F'_{\partial_t u}\eta_u, \eta_\theta\right)$ in (2.15) is of order γ and does not require additional estimates).

Computing now the d-dimensional trace of the operator $\mathcal{B}_s$ in terms of the eigenvalues λ_i, using asymptotics (2.1) for them and arguing as in the end of Theorem 2.1 we derive the improved estimate (2.19). End of the proof.

Corollary 2.1. *Let the assumptions of Theorem 1.1 hold and let, in addition,* (2.18) *be satisfied with the exponent $s > \frac{1}{\kappa} - \frac{1}{2}$. Then, the dimension of the attractor $\mathcal{A}$ possesses the following upper bound:*

$$\dim_F(\mathcal{A}, \mathrm{E}) \le C\gamma^{-1} \quad \text{as } \gamma \to 0, \tag{2.22}$$

where the constant C is independent of γ.

Let us now introduce the class $\mathbb{S}$ of smoothing nonlinearities F.

Definition 2.2. A nonlinear operator $F : \mathrm{E} \to \mathrm{H}$ belongs to the class $\mathbb{S} = \mathbb{S}(C_{k,m})$ if, for every $m \in \mathbb{R}_+$, this operator belongs to $C^\infty(\mathrm{E}^{-m}, \mathrm{H}^m)$ and the following estimates valid, for every $k \in \mathbb{N} \cup \{0\}$:

$$\|F\|_{C^k_b(\mathrm{E}^{-m}, \mathrm{H}^m)} \le C_{k,m}, \tag{2.23}$$

for appropriate constants $C_{k,m}$.

Corollary 2.2. *Let the eigenvalues of the operator* A *satisfy condition* (2.1) *and let the nonlinearity F belong to the class $\mathbb{S}$. Then, the fractal dimension of the corresponding global attractor $\mathcal{A}$ associated with equation* (1.1) *possesses the following upper estimate:*

$$\dim_F(\mathcal{A}, \mathrm{E}) \le C\frac{1}{\gamma}, \tag{2.24}$$

where the constant C depend on constants $C_{1,m}$ (defined in (2.23)*) and on constants C_1, C_2 and κ (defined in* (2.1)*), but are independent of γ.*

Remark 2.2. In Section 4, we will show that even in the case of extremely regular nonlinearities $F \in \mathbb{S}$, the dimension of the attractor $\mathcal{A}$ indeed may have the rate growth $\sim \gamma^{-1}$ as $\gamma \to 0$. So, estimate (2.24) is indeed sharp with respect to $\gamma \to 0$.

§3 Bifurcations of a homoclinic loop and Lyapunov dimension.

In this Section, we consider bifurcations of a certain type of homoclinic loops; the results will be essentially used in the next Section in order to obtain sharp lower bounds for the fractal dimension of the attractor $\mathcal{A}$ in the class $\mathbb{S}$.

In contrast to the previous Sections, we consider here *finite-dimensional* systems of ODE's, namely, systems of the following form:

$$\dot y = \mathbb{A}y + \mathbb{F}(y), \quad y \in \mathbb{R}^n, \tag{3.1}$$

where the nonlinearity $\mathbb{F}(y)$ belongs to $C^\infty(\mathbb{R}^n, \mathbb{R}^n)$ and satisfies

$$\mathbb{F}(0) = \mathbb{F}'(0) = 0. \tag{3.2}$$

We assume that the matrix $\mathbb{A} \in \mathcal{L}(\mathbb{R}^n, \mathbb{R}^n)$ has only one eigenvalue to the right of the imaginary axis. By scaling the time variable in (3.1) we can always make this eigenvalue equal to 1. The rest of the spectrum consists of m pairs of complex eigenvalues $(-\lambda_1 \pm \omega_1, \dots, -\lambda_m \pm \omega_m)$ and $(n - 2m - 1)$ eigenvalues whose real parts are less than some $-\lambda_{m+1} < -\lambda_m$. Here m is some positive integer such that $2m + 1 \le n$, and $\lambda = (\lambda_1, \cdots, \lambda_m) \in \mathbb{R}^m$ and $\omega = (\omega_1, \cdots, \omega_m) \in \mathbb{R}^m$ are given vectors satisfying the condition

$$0 < \lambda_1 \le \lambda_2 \le \cdots \le \lambda_m < 1, \quad \omega_k > 0, \quad k = 1, \cdots, m. \tag{3.3}$$

We assume that the matrix $\mathbb{A}$ can be brought to the following form by a linear transformation of coordinates:

$$\mathbb{A} := \begin{pmatrix} 1 & 0 & 0 & \cdots & 0 \\ 0 & R_1 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & \cdots & 0 & R_m & 0 \\ 0 & \cdots & 0 & 0 & A \end{pmatrix}, \quad R_k := \begin{pmatrix} -\lambda_k & \omega_k \\ -\omega_k & -\lambda_k \end{pmatrix}, \tag{3.4}$$

where the matrix $A \in \mathcal{L}(\mathbb{R}^{n-2m-1}, \mathbb{R}^{n-2m-1})$ satisfies the spectral assumption:

$$\operatorname{Re}\sigma(A) \le -\lambda_{m+1} \quad \text{with} \quad \lambda_{m+1} > \lambda_m. \tag{3.5}$$

Such transformation can always be done when all ω's are different. We, however, prefer not to make this assumption. Instead, we simply assume that the matrix $\mathbb{A}$ is in the form (3.4).

Accordingly, denoting $y = (z, (u_1, v_1), \cdots, (u_m, v_m), w)$, system (3.1) is written in the following form :

$$
(3.6) \qquad \begin{cases} \dot{z} = z + \dots, \\ \dot{u}_k = -\lambda_k u_k + \omega_k v_k + \dots, \\ \dot{v}_k = -\omega_k u_k - \lambda_k v_k + \dots, \qquad\qquad k = 1, \dots, m, \\ \dot{w} = Aw + \dots, \end{cases}
$$

where the dots in (3.6) stand for the nonlinearities, i.e. for the terms vanishing at the origin along with their first derivatives.

By construction, system (3.1) has a hyperbolic equilibrium at the origin O : $y = 0$. Moreover, the unstable manifold $W^u(O)$ is one-dimensional here, and it is tangent to the z-axis at the origin. $W^u \backslash O$ consists of two orbits (the separatrices) which leave the origin at $t = -\infty$ in the opposite directions. We assume that one of the separatrices which leaves O towards positive z (we denote this separatrix as Γ) returns to the origin as $t \to +\infty$, thus it forms a homoclinic loop.

The system under consideration has an $(n-2m-1)$-dimensional smooth invariant strong-stable manifold W^{ss} which is tangent at O to the w-space and which consists of all orbits which tend to O faster than $e^{-\lambda_m t}$ (see e.g. [ShShTCh98]). We assume that *the homoclinic loop* Γ *belongs to* W^{ss}, i.e. it enters O being tangent to the w-space. Note that this is a bifurcation of codimension $(2m + 1)$, so we study here a problem which is, at large m, very degenerate from the point of view of the conventional bifurcation theory. However, the homoclinic loops of these type are quite typical for integrable equations with damping (see e.g. system (4.7) in the next Section).

Finally, we assume that

$$
(3.7) \qquad 2\lambda_1 + 2\lambda_2 + \cdots + 2\lambda_m < 1, \quad 2\lambda_1 + 2\lambda_2 + \cdots + 2\lambda_m + \lambda_{m+1} > 1.
$$

Remark 3.1. We note that inequalities (3.5) and (3.7) imply that the flow defined by (3.6) contracts $(2m+2)$-dimensional volumes near O while $(2m+1)$-dimensional volumes are not contracted. Thus, the Lyapunov dimension $\dim_L(\mathbb{A})$ of system (3.1) at the origin possesses the estimates (see e.g. [Tem97]):

$$
2m + 1 < \dim_L(\mathbb{A}_0) < 2m + 2.
$$

The main result of this Section is the following theorem.

Theorem 3.1. *Let the above assumptions hold. Let* $\mathcal{R} := \{\mu_1, \cdots, \mu_{2m}\}$ *be an arbitrary set of* $2m$ *non-zero complex numbers such that if* μ *belongs to the set* $\mathcal{R}$, *then its complex-conjugate* $\bar{\mu}$ *belongs to* $\mathcal{R}$ *as well. Then, by an arbitrarily small* C^∞*-perturbation of system* (3.1), *a periodic orbit with* $2m$ *multipliers equal to* $\mu_1, \dots, \mu_{2m}$ *and with the rest of the multipliers inside the unit circle can be born from the homoclinic loop, i.e. for an arbitrarily small neighborhood* $\mathbb{V}$ *of the homoclinic loop* Γ, *for every* $\varepsilon > 0$ *and every* $r \in \mathbb{N}$, *there exists a* C^∞*-function* $\mathbb{G}_\varepsilon$ *satisfying the inequality*

$$
\|\mathbb{G}_\varepsilon - (\mathbb{A}_0 + \mathbb{F})\|_{C^k(\mathbb{R}^n, \mathbb{R}^n)} \le \varepsilon,
$$

such that the perturbed system $\dot{y} = \mathbb{G}_\varepsilon(y)$ possesses a periodic orbit of the type described above, which lies in $\mathbb{V}$.

Proof. Let us first locally straighten invariant manifolds W^u, W^s and $W^{ss} \subset W^s$, i.e. we make a coordinate transformation in a small neighborhood of the origin such that the system takes, locally, the form

$$\text{(3.8)} \qquad \begin{cases} \dot{z} = z(1 + p(y)), \\ \dot{u}_k = -\lambda_k u_k + \omega_k v_k + f_k(y) \cdot (u, v, w), \\ \dot{v}_k = -\omega_k u_k - \lambda_k v_k + g_k(y) \cdot (u, v, w), \qquad k = 1, \ldots, m, \\ \dot{w} = (A + q(y))w \end{cases}$$

where the functions $f_k(y)$, $g_k(y)$, $p(y)$, $q(y)$ vanish at the origin. In these coordinates we have $W^u_{loc} = \{(u_k, v_k) = 0 \ (k = 1, \ldots, m), w = 0\}$, $W^s_{loc} = \{z = 0\}$, $W^{ss}_{loc} = \{z = 0, (u_k, v_k) = 0 \ (k = 1, \ldots, m)\}$, so the invariant manifolds are straightened indeed. When the system is brought to this form, we can freely change the characteristic exponents (i.e. $-\lambda_k \pm i\omega_k$ and the eigenvalues of A) by localized small perturbations, without destroying the homoclinic loop. Indeed, we may arbitrarily add small localized perturbations to the coefficients λ_k, ω_k and A in (3.8), and this will not move the local invariant manifolds W^u_{loc}, W^s_{loc}, W^{ss}_{loc}. Thus, by applying perturbations of this kind, we will still have a homoclinic loop which enters O lying in W^{ss}_{loc}. So, we may always assume that

$$\text{(3.9)} \qquad \lambda_1 < \lambda_2 < \cdots < \lambda_m < \lambda_{m+1}.$$

Moreover, we may always achieve by an arbitrarily small such perturbation that the set of characteristic exponents is non-resonant. After that is done, Sternberg's theorem (see e.g. [KaH95]) is applied which means that we can make a smooth coordinate transformation which makes the system linear in a small neighborhood of O, i.e. the system takes, locally, the form

$$\text{(3.10)} \qquad \begin{cases} \dot{z} = z, \\ \dot{u}_k = -\lambda_k u_k + \omega_k v_k, \\ \dot{v}_k = -\omega_k u_k - \lambda_k v_k, \qquad k = 1, \ldots, m, \\ \dot{w} = Aw. \end{cases}$$

In other words, after the above transformations, our equation reads as

$$\text{(3.11)} \qquad \dot{y} = \mathbb{A}(\omega)y + \mathbb{F}(y), \quad y := (z, u_1, v_1, \cdots, u_n, v_n, w) \in \mathbb{R}^n,$$

with the matrix $\mathbb{A}$ given by (3.4) (from now on, we fix λ's satisfying (3.7) and (3.9), but we will vary the values of $\omega_1, \ldots, \omega_m$, therefore we indicate the dependence of $\mathbb{A}$ on ω explicitly). The smooth nonlinear function $\mathbb{F}$ vanishes in some neighbourhood $\mathcal{O}$ of the origin

$$\text{(3.12)} \qquad \mathbb{F}(y) \equiv 0 \quad \text{for } y \in \mathcal{O}.$$

Thus, by construction, the intersection of the homoclinic loop Γ with $\mathcal{O}$ consists of two pieces. The first piece, corresponding to the large negative times, coincides with the positive local z-axis, and the second piece, corresponding to the large positive t, lies in the w-space.

Solution of (3.10) which starts in $\mathcal{O}$ at a point $(z^0, u_1^0, v_1^0, \dots, u_m^0, v_m^0, w^0)$ is written as

$$\begin{aligned} z(t) &= z^0 e^t, \\ u_k(t) &= e^{-\lambda_k t}(u_k^0 \cos\omega_k t + v_k^0 \sin\omega_k t), \\ v_k(t) &= e^{-\lambda_k t}(-u_k^0 \sin\omega_k t + v_k^0 \cos\omega_k t), \\ w(t) &= e^{At} w^0. \end{aligned} \tag{3.13}$$

We take some small $d > 0$ and consider two cross-sections to the homoclinic loop: $\Pi^{out} = \{z = d\}$ and $\Pi^{in} = \{\|w\|_A = d\}$ where the metric $\|w\|_A$ in the w-space $\mathbb{R}^{n-2m-1}$ is defined as follows:

$$\|w\|_A^2 := \int_0^\infty \|e^{At} w\|^2 \, dt$$

and $\|\cdot\|$ is a standard norm in $\mathbb{R}^{n-2m-1}$. Then, obviously,

$$\frac{d}{dt}(\|w\|_A^2) = -2\|w\|^2 < 0,$$

and, consequently, every nonzero solution $w = w(t)$ of equation $\dot{w} = Aw$ intersects transversely the ellipsoid $\|w\|_A = d$ at a unique point. Therefore, the Poincare section Π^{in} is, indeed, well defined. Let also w_0 correspond to the intersection point of the homoclinic loop with Π^{in}, and let $\alpha \in \mathbb{R}^{n-2m-2}$ be local coordinates on the ellipsoid $\|w\|_A = d$ near w_0, i.e. there is a smooth function $\mathcal{W} : \mathbb{R}^{n-2m-2} \to \mathbb{R}^{n-2m-1}$ such that $\|\mathcal{W}(\alpha)\|_A \equiv d$, $\mathcal{W}(0) = w_0$ and $\mathcal{W}'(0) = \mathrm{Id}$. We introduce the local coordinates for $M \in \Pi^{in}$ and $\bar{M} \in \Pi^{out}$ as follows

$$\begin{aligned} M(Z, u_1, v_1, \cdots, u_m, v_m, \alpha) &:= (d \cdot Z, u_1, v_1, \cdots, u_m, v_m, \mathcal{W}(\alpha)), \\ \bar{M}(\bar{u}_1, \bar{v}_1, \cdots, \bar{u}_m, \bar{v}_m, \bar{w}) &:= (d, \bar{u}_1, \bar{v}_1, \cdots, \bar{u}_m, \bar{v}_m, \bar{w}) \end{aligned}$$

According to (3.13), the orbit of a point $M \in \Pi^{in}$ with $Z > 0$ reaches Π^{out} at the moment of time $t = -\ln Z$, and the intersection of the orbit with Π^{out} is the point

$$\bar{M} := T_\omega^{loc}(M) := \begin{pmatrix} Z^{\lambda_1}(u_1 \cos\omega_1 \ln Z - v_1 \sin\omega_1 \ln Z) \\ Z^{\lambda_1}(u_1 \sin\omega_1 \ln Z + v_1 \cos\omega_1 \ln Z) \\ \cdots \\ Z^{\lambda_k}(u_k \cos\omega_k \ln Z - v_k \sin\omega_k \ln Z) \\ Z^{\lambda_k}(u_k \sin\omega_k \ln Z + v_k \cos\omega_k \ln Z) \\ Z^{-A}\mathcal{W}(\alpha) \end{pmatrix} \tag{3.14}$$

which defines the local Poincare map $T_\omega^{loc} : \Pi^{in} \cap \{Z > 0\} \to \Pi^{out}$.

Analogously, the orbits starting on Π^{out} close to the origin follow the homoclinic loop, so they come to the cross-section Π^{in} in finite time. These orbits define a global Poincaré map $T_0^{glo} : \Pi^{out} \to \Pi^{in}$ which is a diffeomorphism (since it is defined by orbits of a smooth flow in a finite time and since the trajectories intersect Π^{in} transversely). Thus, the linear operator

$$\mathcal{T}_0 := \frac{d}{d\bar{M}} T_0^{glo}(0) \tag{3.15}$$

is invertible and, due to our choice of coordinates in Π^{in} and Π^{out},

$$T_0^{glo}(0) = 0. \tag{3.16}$$

Moreover, without loss of generality we assume that $\mathcal{T}_0 \in \mathcal{L}(\mathbb{R}^{n-2m-2}, \mathbb{R}^{n-2m-2})$ can be represented as follows:

$$\mathcal{T}_0 = L_0 \cdot U_0, \tag{3.17}$$

where U_0 and L_0 are upper- and lower-triangular matrices respectively:

$$L_0 = \begin{pmatrix} L^0_{11} & 0 & \cdots & & \\ L^0_{21} & L^0_{22} & 0 & \cdots & \\ L^0_{31} & L^0_{32} & L^0_{33} & 0 & \cdots \\ & & & \ddots & \\ \cdots & \cdots & \cdots & & \end{pmatrix}, \qquad U_0 = \begin{pmatrix} U^0_{11} & U^0_{12} & U^0_{13} & \cdots \\ 0 & U^0_{22} & U^0_{23} & \cdots \\ \cdots & 0 & U^0_{33} & \cdots \\ & & 0 & \ddots \\ \cdots & \cdots & \cdots & \end{pmatrix}. \tag{3.18}$$

Indeed, decomposition (3.17) is well known for generic invertible matrices $\mathcal{T}_0$ (and can be obtained e.g. via classical Gauss diagonalization procedure). If $\mathcal{T}_0$ is not generic, we can always put it in a general position by an arbitrarily small perturbation of (3.11) which is localized *outside* the d-neighbourhood of the origin and *preserves* the homoclinic loop (using the standard flow-box technique). Note that

$$U^0_{ii} \neq 0, \quad L^0_{ii} \neq 0, \text{ for } i = 1, \cdots, n-2m-2, \tag{3.19}$$

since $\mathcal{T}_0$ is invertible.

We now consider an $(m+n-1)$-parameter family of small smooth perturbations of the system (3.11), namely for every $\omega \in \mathbb{R}^m$ which is sufficiently close to the original vector $\omega := \omega^0$ and for every sufficiently small $\theta \in \mathbb{R}^{n-1}$, we consider the following family of equations:

$$\dot{y} = \mathbb{A}(\omega)y + \mathbb{F}_{\theta,\omega}(y). \tag{3.20}$$

We assume that the function $\mathbb{F}_{\theta,\omega}$ is smooth with respect to all variables and satisfies the assumptions

$$\mathbb{F}_{\theta,\omega}(y) \equiv 0 \text{ for } y \in \mathcal{O}, \text{ and } \mathbb{F}_{0,0}(y) \equiv \mathbb{F}(y), \tag{3.21}$$

where $\mathbb{F}(y)$ is defined in (3.11). Then, for sufficiently small θ and $(\omega - \omega^0)$, the global Poincare map $T^{glo}_{\theta,\omega} : \Pi^{out} \to \Pi^{in}$ is well defined and smooth. We assume that the perturbation (3.20) is such that this global Poincare map is written, in a small neighborhood of the origin in Π^{out}, as follows:

$$T^{glo}_{\theta,\omega}(\bar{M}) = \theta + T^{glo}_0(\bar{M}), \tag{3.22}$$

i.e. the only effect of the perturbation in the nonlinearity $\mathbb{F}$ is an additive term in the global map (see (3.16)). Obviously, such a family of perturbations exists (one can construct it by the flow-box technique). Since the global map is insensitive to changes in ω we will further use the notation T^{glo}_θ.

It is obvious, that for every frequency vector ω which is sufficiently close to ω^0 and for every $M \in \Pi^{in} \cap \{Z > 0\}$ which is sufficiently close to 0 (in our local coordinates on Π^{in}) there exists a perturbation parameter θ such that system (3.20) possesses a periodic orbit which intersects with Π^{in} at the given point M. Indeed, note that, due to our construction, fixed points M with $Z > 0$ of the first-return map

$$T_{\theta,\omega}(M) := T^{glo}_\theta(T^{loc}_\omega(M)), \quad T_{\theta,\omega} : \Pi^{in} \to \Pi^{in}, \tag{3.23}$$

correspond to periodic orbits of the system (3.20). Thus, we must find the value of θ for which the given point M is the fixed point of map (3.23). It remains to note that the fixed point equation $T_{\theta,\omega}M = M$ recasts, by virtue of (3.22),(3.23), as the following relation

$$\theta = M - T^{glo}_0(T^{loc}_\omega(M)) \tag{3.24}$$

which indeed defines θ uniquely, given ω and M.

Our next step is to compute the multipliers of the periodic orbit in dependence on ω and M. By definition, these multipliers are the eigenvalues of the derivative with respect to M of the first-return map (3.23):

$$P(\omega, M) := \frac{d}{dM} T_{\theta,\omega}(M)\big|_{\theta=\theta(\omega,M)} = \mathcal{T}_{\omega,M} \circ \mathcal{Z}(\omega, M) \tag{3.25}$$

where

$$\mathcal{T}_{\omega,M} := \frac{d}{d\bar{M}} T^{glo}_\theta(\bar{M})\big|_{\theta=\theta(\omega,M), \bar{M}=T^{loc}_\omega(M)}, \quad \mathcal{Z}(\omega, M) := \frac{d}{dM} T^{loc}_\omega(M). \tag{3.26}$$

As M tends to the point $(\mathcal{Z}, u_1, v_1, \cdots, u_m, v_m, \alpha) = 0$ (this is the point of intersection of the homoclinic loop Γ with the cross-section Π^{in} at $\theta = 0$), we have $\theta \to 0$, by virtue of (3.24),(3.14),(3.16). Thus, as $\omega \to \omega^0$, $M \to 0$, the matrix $\mathcal{T}_{\omega,M}$ tends to the matrix $\mathcal{T}_0$ defined by (3.15).

On the other hand, differentiating (3.14) with respect to M, we obtain

$$\mathcal{Z}(\omega, M) = \begin{pmatrix} \begin{matrix} Z^{\lambda_1-1}\cos(\omega_1 \ln Z + \varphi_1)\rho_1 \\ Z^{\lambda_1-1}\sin(\omega_1 \ln Z + \varphi_1)\rho_1 \end{matrix} & \mathcal{R}_1(Z) & 0 & \dots & 0 \\ \begin{matrix} Z^{\lambda_2-1}\cos(\omega_2 \ln Z + \varphi_2)\rho_2 \\ Z^{\lambda_2-1}\sin(\omega_2 \ln Z + \varphi_2)\rho_2 \end{matrix} & 0 & \mathcal{R}_2(Z) & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ -AZ^{A-I} & 0 & 0 & \dots & Z^{-A} \end{pmatrix}, \tag{3.27}$$

where we denote

$$\mathcal{R}_k(Z, \omega) = Z^{\lambda_k} \begin{pmatrix} \cos\omega_k \ln Z & -\sin\omega_k \ln Z \\ \sin\omega_k \ln Z & \cos\omega_k \ln Z \end{pmatrix} \tag{3.28}$$

and use, notationally, polar coordinates (ρ_k, φ_k) instead of (u_k, v_k) in the following way:

$$u_k = \frac{\rho_k}{r_k}\cos(\varphi_k - \psi_k), \quad v_k = \frac{\rho_k}{r_k}\sin(\varphi_k - \psi_k), \tag{3.29}$$

with

$$r_k := (\lambda_k^2 + \omega_k^2)^{1/2}, \quad \cos\psi_k = \lambda_k/r_k \quad \text{and} \quad \sin\psi_k = \omega_k/r_k. \tag{3.30}$$

In the sequel, we will study the eigenvalues of the matrix $P(\omega, M)$ defined by (3.25) for (ω, M) of some special form only. To be more precise, we fix some positive numbers β_k, $k = 1, \cdots, m$, such that

$$\begin{aligned} 1 - 2\lambda_1 - \cdots - 2\lambda_m > \beta_1 > \beta_2 > \cdots > \beta_m > \\ > \beta_1 - \min\{\lambda_2 - \lambda_1, \lambda_3 - \lambda_2, \dots, \lambda_m - \lambda_{m-1}, \lambda_{m+1} - \lambda_m\}, \end{aligned} \tag{3.31}$$

(such numbers exist due to assumption (3.7)). Then, we fix

$$\rho_k = Z^{\beta_k}. \tag{3.32}$$

Moreover, we consider the perturbations of the frequency vector ω^0 in the form

$$\omega(\bar\omega) := \omega^0 + (\ln Z)^{-1}\bar\omega, \tag{3.33}$$

where $\bar\omega \in [-\pi, 3\pi]^m$. Then, for every small positive $Z \ll 1$, every $\bar\omega \in [-\pi, 3\pi]^m$ and every $\phi = (\phi_1, \cdots, \phi_m) \in [-\pi, 3\pi]^m$, the point $M = M(Z, \bar\omega, \phi)$ is defined as follows:

$$\begin{aligned} M(Z, \bar\omega, \phi) := y = (d\cdot Z, \frac{\rho_1}{r_1}\cos(\phi_1 - \psi_1), \frac{\rho_1}{r_1}\sin(\phi_1 - \psi_1), \cdots \\ \cdots, \frac{\rho_m}{r_m}\cos(\phi_m - \psi_m), \frac{\rho_m}{r_m}\sin(\phi_m - \psi_m), \mathcal{W}(0)), \end{aligned} \tag{3.34}$$

where the parameters $\rho_k = \rho_k(Z)$, $\omega = \omega(\bar\omega)$, $r_k = r_k(\bar\omega)$ and $\psi_k = \psi_k(\bar\omega)$ are defined by (3.32), (3.33) and (3.30).

Thus, we consider finally the $(2m + 1)$-parameter family of perturbations of equation (3.11) which corresponds to the choice of M of the form (3.34) and ω in the form (3.33) with small positive $Z \ll 1$ and arbitrary $\bar\omega, \phi \in [-\pi, 3\pi]^m$ and study the matrix (3.25) only for such (ω, M). In order to simplify the notations, we write in the sequel $P(Z, \bar\omega, \phi)$ instead of $P(\omega(\bar\omega), M(Z, \bar\omega, \phi))$, $\mathcal{Z}(Z, \bar\omega, \phi)$ instead of $\mathcal{Z}(\omega(\bar\omega), M(Z, \bar\omega, \phi))$ and so on. It is obvious that our family of perturbations is, indeed, arbitrarily small when $Z \to +0$.

The next lemma gives the principle part of the asymptotic expansions of the coefficients of the characteristic polynomial for the matrix $P(Z, \bar\omega, \phi)$ as $Z \to +0$.

Lemma 3.2. *Let*

$$\begin{aligned} \mathbb{P}_{Z,\bar\omega,\phi}(\mu) := \det(\mu\,\mathrm{Id} - P(Z, \bar\omega, \phi)) := \\ = \mu^n - \mathcal{M}_1\mu^{n-1} + \mathcal{M}_2\mu^{n-2} + \cdots + (-1)^n\mathcal{M}_n \end{aligned} \tag{3.35}$$

be the characteristic polynomial of the matrix $P(Z,\bar\omega,\phi)$ *defined by* (3.25). *Then the following formulas are valid for the coefficients* $\mathcal{M}_k$ *of this polynomial:*

$$\mathcal{M}_{2k-1}(Z,\bar\omega,\phi)=\left(\prod_{j=1}^{2k-1}L^0_{jj}\right)\cdot\left(\prod_{j=1}^{2k-2}U^0_{jj}\right)Z^{-1+2\lambda_1+\cdots+2\lambda_{k-1}+\lambda_k+\beta_k}\times$$
$$\times\ \Big[U^0_{2k-1,2k-1}\cos(\omega^0_k\ln Z+\varphi_k+\bar\omega_k)+$$
$$+U^0_{2k-1,2k}\sin(\omega^0_k\ln Z+\varphi_k+\bar\omega_k)+\mathbb{M}_{2k-1}(Z,\bar\omega,\phi)\Big],\tag{3.36}$$
$$\mathcal{M}_{2k}(Z,\bar\omega,\phi)=\left(\prod_{j=1}^{2k-1}L^0_{jj}\right)\cdot\left(\prod_{j=1}^{2k}U^0_{jj}\right)Z^{-1+2\lambda_1+\cdots+2\lambda_k+\beta_k}\times$$
$$\times\Big[-L^0_{2k,2k}\sin\varphi_k+L^0_{2k+1,2k}\cos\varphi_k+\mathbb{M}_{2k}(Z,\bar\omega,\phi)\Big]$$

for $k=1,\cdots,m$, *and*

$$\mathcal{M}_k=\mathbb{M}_k(Z,\bar\omega,\phi)\tag{3.37}$$

for $k>2m$. *Here* L^0_{ij} *and* U^0_{ij} *are the entries of the lower- and, respectively, upper-triangular matrices defined by* (3.17). *The functions* $\mathbb{M}_k$ *are smooth with respect to* $(\bar\omega,\phi)$ *and* $Z>0$, *and they tend to zero, along with their derivatives with respect to* $(\bar\omega,\phi)$, *as* $Z\to+0$.

Proof. We recall that the matrix $\mathcal{T}_{Z,\bar\omega,\phi}:=\mathcal{T}_{(\omega(\bar\omega),M(Z,\bar\omega,\phi))}$ in (3.25) is close to the matrix $\mathcal{T}_0$, hence it can be decomposed, analogously to (3.17):

$$\mathcal{T}_{Z,\bar\omega,\phi}=L_{Z,\bar\omega,\phi}\cdot U_{Z,\bar\omega,\phi},\tag{3.38}$$

where U and L are upper- and lower-triangular matrices respectively:

$$L=\begin{pmatrix}L_{11}&0&\dots&&\\ L_{21}&L_{22}&0&\dots&\\ L_{31}&L_{32}&L_{33}&0&\dots\\ &&&\ddots&\\ \dots&\dots&\dots&&\end{pmatrix},\qquad U=\begin{pmatrix}U_{11}&U_{12}&U_{13}&\dots\\ 0&U_{22}&U_{23}&\dots\\ \dots&0&U_{33}&\dots\\ &&0&\ddots\\ \dots&\dots&\dots&\end{pmatrix}.\tag{3.39}$$

Moreover, the entries $U_{ij}=U_{ij}(Z,\bar\omega,\phi)$ and $L_{ij}=L_{ij}(Z,\bar\omega,\phi)$ are smooth with respect to all arguments and are close to the corresponding entries U^0_{ij} and L^0_{ij} as $Z\ll1$.

We now note that the matrix

$$C=C(Z,\bar\omega,\phi):=U\cdot\mathcal{Z}(Z,\bar\omega,\phi)\cdot L\tag{3.40}$$

is similar to $P(Z,\bar\omega,\phi)=L\cdot U\cdot Z(Z,\bar\omega,\phi)$ and, consequently, these two matrices have the same characteristic polynomials. So, we compute below the characteristic polynomial of the matrix C defined by (3.40). In order to do so, we recall that assumptions (3.9), (3.31) and (3.32) imply the following inequalities:

$$
\begin{aligned}
&Z \ll \rho_1 \ll \rho_2 \ll \cdots \ll \rho_m \ll 1, \\
&Z^{\lambda_1}\rho_1 \gg Z^{\lambda_2}\rho_2 \gg \ldots Z^{\lambda_m}\rho_m \gg Z^{\lambda_{m+1}}, \\
&Z^{\lambda_1} \gg Z^{\lambda_2} \gg \ldots Z^{\lambda_m} \gg Z^{\lambda_{m+1}},
\end{aligned} \tag{3.41}
$$

if $Z \ll 1$. Since the matrices U and L are upper- and lower-triangular, the entries of the matrix C are given by the following formula

$$
C_{ij} = \sum_{m=i}^{n}\sum_{k=j}^{n} \mathcal{Z}_{mk} U_{im} L_{kj}. \tag{3.42}
$$

Computing by (3.27) and (3.42), and using (3.41), it is easy to verify that the matrix C defined by (3.40) is estimated as follows:

$$
\begin{pmatrix}
O(Z^{\lambda_1-1}\rho_1) & O(Z^{\lambda_1}) & O(Z^{\lambda_1}) & O(Z^{\lambda_2}) & O(Z^{\lambda_2}) & O(Z^{\lambda_3}) & \ldots \\
O(Z^{\lambda_1-1}\rho_1) & O(Z^{\lambda_1}) & O(Z^{\lambda_1}) & O(Z^{\lambda_2}) & O(Z^{\lambda_2}) & O(Z^{\lambda_3}) & \ldots \\
O(Z^{\lambda_2-1}\rho_2) & O(Z^{\lambda_2}) & O(Z^{\lambda_2}) & O(Z^{\lambda_2}) & O(Z^{\lambda_2}) & O(Z^{\lambda_3}) & \ldots \\
O(Z^{\lambda_2-1}\rho_2) & O(Z^{\lambda_2}) & O(Z^{\lambda_2}) & O(Z^{\lambda_2}) & O(Z^{\lambda_2}) & O(Z^{\lambda_3}) & \ldots \\
\vdots & \vdots & \vdots & \vdots & \vdots & \ddots &
\end{pmatrix}, \tag{3.43}
$$

i.e. its entries are estimated as follows:

$$
C_{i1} = O(Z^{\lambda_{k(i)}-1}\rho_{k(i)}), \quad C_{ij} = O(Z^{\lambda_{s(i,j)}}) \quad (j \geq 2), \tag{3.44}
$$

where we denote

$$
k(i) = \begin{cases}
1 & \text{at } i = 1, 2, \\
2 & \text{at } i = 3, 4, \\
\vdots & \\
m & \text{at } i = 2m-1, 2m, \\
m+1 & \text{at } i > 2m,
\end{cases} \tag{3.45}
$$

and

$$
s(i,j) = \begin{cases}
m+1 & \text{at } i > 2m \\
\text{or (at } i \leq 2m) & \\
k(i) & \text{at } j = 2, \ldots, 2k(i)+1, \\
k(i)+1 & \text{at } j = 2k(i)+2, 2k(i)+3, \\
\vdots & \\
m & \text{at } j = 2m, 2m+1, \\
m+1 & \text{at } j > 2m+1.
\end{cases} \tag{3.46}
$$

We also denote here $\rho_{m+1} = 1$.

We recall now that the p-th coefficient $\mathcal{M}_p = \mathcal{M}_p(Z, \bar{\omega}, \phi)$, $p = 1, \cdots, n$ of the characteristic polynomial (3.35) can be represented as a sum of all main (or diagonal) minors of order p of the matrix C defined by (3.40), i.e.

$$\mathcal{M}_p = \sum_{1 \le i_1 < i_2 < \cdots < \cdots < i_p \le n} M_{i_1, \dots, i_p}(C), \tag{3.47}$$

where the minor $M_{i_1,\dots,i_p}(C)$ is the determinant of the matrix obtained as the intersection of the raws with the numbers $i_1, \dots, i_p$ and the columns with the same numbers.

Our task now is to show that the major contribution to $\mathcal{M}_p$ at $Z \ll 1$ is given by the minor $M_{1,2,\cdots,p}$. Indeed, it follows from (3.44) that all the entries C_{ij} with $i > 1$ vanish at $Z = 0$ and, consequently, all the diagonal minors $M_{i_1,\dots,i_p}(C)$ with $i_1 > 1$ tend to zero as $Z \to 0$. For the minors $M_{i_1=1,i_2,\dots,i_p}(C)$ we use the following formula:

$$M_{i_1=1,i_2,\dots,i_p} = Z^{\lambda_{k_1}+\lambda_{k_2}+\cdots+\lambda_{k_p}-1} \rho_{k_p} \cdot \det \mathcal{C}_{1,i_2,\dots,i_p} \tag{3.48}$$

where we denote $k_q \equiv k(i_q)$ (see (3.45)), and

$$\mathcal{C}_{1,i_2,\dots,i_p} = \begin{pmatrix} Z^{1-\lambda_1} \rho_{k_p}^{-1} C_{11} & Z^{-\lambda_1} C_{1i_2} & \dots & Z^{-\lambda_1} C_{1i_p} \\ Z^{1-\lambda_{k_2}} \rho_{k_p}^{-1} C_{i_2 1} & Z^{-\lambda_{k_2}} C_{i_2 i_2} & \dots & Z^{-\lambda_{k_2}} C_{i_2 i_p} \\ \vdots & \vdots & \ddots & \vdots \\ Z^{1-\lambda_{k_p}} \rho_{k_p}^{-1} C_{i_p 1} & Z^{-\lambda_{k_p}} C_{i_p i_2} & \dots & Z^{-\lambda_{k_p}} C_{i_p i_p} \end{pmatrix}. \tag{3.49}$$

By (3.41) and (3.43), all the entries of the matrix $\mathcal{C}$ are bounded from above, so we have

$$M_{1,i_2,\dots,i_p}(C) = O(Z^{\lambda_{k_1}+\lambda_{k_2}+\cdots+\lambda_{k_p}-1} \rho_{k_p}). \tag{3.50}$$

If $p \ge 2m+1$, this estimate gives us

$$M_{1,i_2,\dots,i_p}(C) = O(Z^{2\lambda_1+2\lambda_2+\cdots+2\lambda_m+\lambda_{m+1}-1}), \tag{3.51}$$

so, by virtue of our assumption (3.7), all the diagonal minors of order $2m+1$ and larger tend to zero as $Z \to +0$, which proves (3.37).

Let us now consider the case $p \le 2m$. We note that when i decreases at least on 2, the corresponding value of $\lambda_{k(i)}$ will also decrease. Thus, it follows from (3.41) and (3.50) that the main contribution to the coefficient $\mathcal{M}_p$ ($p \le 2m$) is given by the minor $M_{1,2,\dots,p}(C)$ in case p is even, and by the two minors $M_{1,2,\dots,p-1,p}(C)$ and $M_{1,2,\dots,p-1,p+1}(C)$ in case p is odd (and $p > 1$). Moreover, we claim that

$$\mathcal{M}_{1,2,\cdots,2(l-1),2l} = Z^{-1+2\lambda_1+\cdots+2\lambda_{l-2}+2\lambda_{l-1}+\lambda_l+\beta_l} O(Z^{\varepsilon}), \tag{3.52}$$

for some $\varepsilon > 0$. Indeed, according to (3.31),(3.32), (3.45),

$$\rho_{k(i)} \rho_l^{-1} = O(Z^{\beta_{k(i)}-\beta_l}) \ll 1 \;\text{ and }\; Z^{-\lambda_{k(i)}} C_{i,2l} = O(Z^{\lambda_l-\lambda_{k(i)}}) \ll 1,$$

for $i \le 2(l-1)$. Consequently, the matrix $\mathcal{C}_{1,\cdots,2(l-1),2l}(C)$ defined via (3.49) can be rewritten as follows:

$$\mathcal{C}_{1,\cdots,2(l-1),2l} = \begin{pmatrix} 0 & O(1) & \cdots & O(1) & 0 \\ 0 & O(1) & \cdots & O(1) & 0 \\ \vdots & \vdots & \cdots & \vdots & \vdots \\ 0 & O(1) & \cdots & O(1) & 0 \\ O(1) & O(1) & \cdots & O(1) & O(1) \end{pmatrix} + O(Z^{\varepsilon}),$$

which implies (3.59), since the determinant of the matrix in the right-hand side of the last formula is, obviously, zero. Thus, we have proved that

$$(3.53)\qquad \begin{aligned} \mathcal{M}_{2k-1} &= M_{1,\cdots,2k-1}(C) + Z^{-1+2\lambda_1+\cdots+2\lambda_{k-1}+\lambda_k+\beta_k} O(Z^{\varepsilon}), \\ \mathcal{M}_{2k} &= M_{1,2,\cdots,2k}(C) + Z^{-1+2\lambda_1+\cdots+2\lambda_k+\beta_k} O(Z^{\varepsilon}) \quad (k = 1, \cdots, m) \end{aligned}$$

for some small positive constant $\varepsilon > 0$. It remains to compute the determinants $M_{1,\cdots,p}$ for $p = 1, \cdots, 2m$.

To this end, according to (3.41) and (3.44), we rewrite the formula (3.49) for $\mathcal{C}_{1,\cdots,2l-1}$ and $\mathcal{C}_{1,\cdots,2l}$ $(l = 1, \cdots, m)$ as follows:

(3.54) $\mathcal{C}_{1,\cdots,2l-1} =$

$$= \begin{pmatrix} 0 & Z^{-\lambda_1}C_{12} & Z^{-\lambda_1}C_{13} & 0 & 0 & 0 & \cdots \\ 0 & Z^{-\lambda_1}C_{22} & Z^{-\lambda_1}C_{23} & 0 & 0 & 0 & \cdots \\ 0 & O(1) & O(1) & Z^{-\lambda_2}C_{34} & Z^{-\lambda_2}C_{35} & 0 & \cdots \\ 0 & O(1) & O(1) & Z^{-\lambda_2}C_{44} & Z^{-\lambda_2}C_{45} & 0 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ Z^{1-\lambda_l}\rho_l^{-1}C_{2l-1,1} & O(1) & \cdots & \cdots & \cdots & \cdots & O(1) \end{pmatrix}$$

$$+ O(Z^{\varepsilon}),$$

and

(3.55) $\mathcal{C}_{1,\cdots,2l} =$

$$\begin{pmatrix} 0 & Z^{-\lambda_1}C_{12} & Z^{-\lambda_1}C_{13} & 0 & 0 & 0 & \cdots \\ 0 & Z^{-\lambda_1}C_{22} & Z^{-\lambda_1}C_{23} & 0 & 0 & 0 & \cdots \\ 0 & O(1) & O(1) & Z^{-\lambda_2}C_{34} & Z^{-\lambda_2}C_{35} & 0 & \cdots \\ 0 & O(1) & O(1) & Z^{-\lambda_2}C_{44} & Z^{-\lambda_2}C_{45} & 0 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ Z^{1-\lambda_l}\rho_l^{-1}C_{2l-1,1} & O(1) & \cdots & \cdots & \cdots & O(1) & Z^{-\lambda_l}C_{2l-1,2l} \\ Z^{1-\lambda_l}\rho_l^{-1}C_{2l,1} & O(1) & \cdots & \cdots & \cdots & O(1) & Z^{-\lambda_l}C_{2l,2l} \end{pmatrix}$$

$$+ O(Z^{\varepsilon}).$$

Since all the entries of $\mathcal{C}_{1,\dots,p}$ are bounded, we obtain from (3.48),(3.54) and (3.55)

$$\text{(3.56)}\quad M_{1,\cdots,2k-1} = C_{2k-1,1} \times \prod_{l=1}^{k-1} \begin{vmatrix} C_{2l-1,2l} & C_{2l-1,2l+1} \\ C_{2l,2l} & C_{2l,2l+1} \end{vmatrix} + \\ + Z^{-1+2\lambda_1+\cdots+2\lambda_{k-1}+\lambda_k+\beta_k} O(Z^{\varepsilon}),$$

and

$$\text{(3.57)}\quad \mathcal{M}_{1,\cdots,2k} = \begin{vmatrix} C_{2k-1,1} & C_{2k-1,2k} \\ C_{2k,1} & C_{2k,2k} \end{vmatrix} \times \prod_{l=1}^{k-1} \begin{vmatrix} C_{2l-1,2l} & C_{2l-1,2l+1} \\ C_{2l,2l} & C_{2l,2l+1} \end{vmatrix} + \\ + Z^{-1+2\lambda_1+\cdots+2\lambda_k+\beta_k} O(Z^{\varepsilon}),$$

for $k = 1, \cdots, m$, where $\varepsilon > 0$ is a small positive number.

Now, it remains to express the right-hand sides of (3.56) and (3.57) in terms of the entries of the matrices $\mathcal{Z}(Z, \bar{\omega}, \phi)$, $U(Z, \bar{\omega}, \phi)$ and $L(Z, \bar{\omega}, \phi)$. One can easily see that, according to (3.27), (3.28), (3.41), (3.42), (3.44),

$$\text{(3.58)}\quad C_{2k-1,1} = L_{11} U_{2k-1,2k-1} \mathcal{Z}_{2k-1,1} + L_{11} U_{2k-1,2k} \mathcal{Z}_{2k,1} + Z^{-1+\lambda_k} \rho_k O(Z^{\varepsilon}) = \\ = Z^{-1+\lambda_k+\beta_k} L_{11}^0 \Big[U_{2k-1,2k-1}^0 \cos(\omega_k^0 \ln Z + \phi_k + \bar{\omega}_k) + \\ + U_{2k-1,2k}^0 \sin(\omega_k^0 \ln Z + \phi_k + \bar{\omega}_k) + o(1) \Big].$$

Analogously,

$$\text{(3.59)}\quad \begin{vmatrix} C_{2l-1,2l} & C_{2l-1,2l+1} \\ C_{2l,2l} & C_{2l,2l+1} \end{vmatrix} = \\ Z^{2\lambda_l} \begin{vmatrix} U_{2l-1,2l-1} & U_{2l-1,2l} \\ 0 & U_{2l,2l} \end{vmatrix} \times \begin{vmatrix} \cos \omega_l \ln Z & -\sin \omega_l \ln Z \\ \sin \omega_l \ln Z & \cos \omega_l \ln Z \end{vmatrix} \times \begin{vmatrix} L_{2l,2l} & 0 \\ L_{2l+1,2l} & L_{2l+1,2l+1} \end{vmatrix} + \\ + Z^{2\lambda_l} O(Z^{\varepsilon}) = Z^{2\lambda_l} \Big[U_{2l-1,2l-1}^0 U_{2l,2l}^0 L_{2l,2l}^0 L_{2l+1,2l+1}^0 \Big] + Z^{2\lambda_l} o(1).$$

And, finally,

$$\text{(3.60)}\quad \begin{vmatrix} C_{2k-1,1} & C_{2k-1,2k} \\ C_{2k,1} & C_{2k,2k} \end{vmatrix} = L_{11} \begin{vmatrix} U_{2k-1,2k-1} & U_{2k-1,2k} \\ 0 & U_{2k,2k} \end{vmatrix} \times \\ \times \left(L_{2k,2k} \begin{vmatrix} \mathcal{Z}_{2k-1,1} & \mathcal{Z}_{2k-1,2k} \\ \mathcal{Z}_{2k,1} & \mathcal{Z}_{2k,2k} \end{vmatrix} + L_{2k+1,2k} \begin{vmatrix} \mathcal{Z}_{2k-1,1} & \mathcal{Z}_{2k-1,2k+1} \\ \mathcal{Z}_{2k,1} & \mathcal{Z}_{2k,2k+1} \end{vmatrix} \right) + \\ + Z^{-1+2\lambda_k+\beta_k} O(Z^{\varepsilon}) = \Big[L_{11}^0 U_{2k-1,2k-1}^0 U_{2k,2k}^0 \Big] Z^{-1+2\lambda_k+\beta_k} \times \\ \times \Big[- L_{2k,2k}^0 \sin \phi_k + L_{2k+1,2k}^0 \cos \phi_k \Big] + Z^{-1+2\lambda_k+\beta_k} o(1).$$

Inserting these formulas into (3.56),(3.57) and (3.53), we obtain expansions (3.36). Lemma 3.2 is proven.

We are now ready to finish the proof of Theorem 3.1. Indeed, let us consider only such sequence of values of $Z \to +0$ for which

$$\left\{\omega_k^0 \frac{\ln Z}{2\pi}\right\} \to 0$$

for all $k = 1, \cdots, m$ (here $\{\cdot\}$ denotes the fractional part). It is easy to see then, that given any fixed values of the coefficients $\mathcal{M}_1, \cdots, \mathcal{M}_{2m}$ of the characteristic polynomial of the derivative matrix $P(Z, \bar{\omega}, \phi)$ of the Poincaré map of the periodic orbit under consideration, the system of equations (3.36) for these coefficients can be resolved with respect to ϕ and $\bar{\omega}$. Moreover, ϕ and $\bar{\omega}$ depend on $\mathcal{M}_1, \cdots, \mathcal{M}_{2m}$ smoothly and have finite limits as $Z \to +0$, along with the derivatives with respect to $(\mathcal{M}_1, \cdots, \mathcal{M}_{2m})$.

Indeed, system (3.36), recast as

(3.61)

$$\begin{cases} U^0_{2k-1,2k-1} \cos(2\pi \left\{\omega_k^0 \frac{\ln Z}{2\pi}\right\} + \phi_k + \bar{\omega}_k) + \\ \qquad + U^0_{2k-1,2k} \sin(2\pi \left\{\omega_k^0 \frac{\ln Z}{2\pi}\right\} + \phi_k + \bar{\omega}_k) = \\ \quad = \mathcal{M}_{2k-1} \left(\prod_{j=1}^{2k-1} L^0_{jj}\right)^{-1} \cdot \left(\prod_{j=1}^{2k-2} U^0_{jj}\right)^{-1} Z^{1-2\lambda_1-\cdots-2\lambda_{k-1}-\lambda_k-\beta_k} - \\ \qquad - \mathbb{M}_{2k-1}(Z, \bar{\omega}, \phi), \\ \\ -L^0_{2k,2k} \sin\phi_k + L^0_{2k+1,2k} \cos\phi_k = \\ \quad = \mathcal{M}_{2k} \left(\prod_{j=1}^{2k-1} L^0_{jj}\right)^{-1} \cdot \left(\prod_{j=1}^{2k} U^0_{jj}\right)^{-1} Z^{1-2\lambda_1-\cdots-2\lambda_k-\beta_k} - \mathbb{M}_{2k}(Z, \bar{\omega}, \phi), \\ \qquad (k = 1, \cdots, m), \end{cases}$$

has a regular limit as $Z \to +0$:

(3.62)
$$\begin{cases} U^0_{2k-1,2k-1} \cos(\phi_k + \bar{\omega}_k) + U^0_{2k-1,2k} \sin(\phi_k + \bar{\omega}_k) = 0, \\ -L^0_{2k,2k} \sin\phi_k + L^0_{2k+1,2k} \cos\phi_k = 0. \end{cases}$$

Here we have used the fact that due to our assumptions (3.7) and (3.31),

$$1 - 2\lambda_1 - \cdots - 2\lambda_k - \beta_k > 0$$

for every $k = 1, \cdots, m$. By (3.19), $U^0_{2k-1,2k-1} \neq 0$ and $L^0_{2k,2k} \neq 0$, so we may resolve the limit system (3.62) as follows:

(3.63)
$$\begin{cases} \phi_k = \arctan \frac{L^0_{2k+1,2k}}{L^0_{2k,2k}} \\ \bar{\omega}_k = \frac{\pi}{2} - \arctan \frac{U^0_{2k-1,2k}}{U^0_{2k-1,2k-1}} - \phi_k. \end{cases}$$

Now, according to the implicit function theorem, we have indeed the functions $\phi(Z, \mathcal{M}_1, \ldots, \mathcal{M}_{2m})$, $\bar{\omega}(Z, \mathcal{M}_1, \ldots, \mathcal{M}_{2m})$, close to those given by (3.63), which satisfy (3.61) (hence, they satisfy (3.36)) at small Z and which depend smoothly on $\mathcal{M}_1, \ldots, \mathcal{M}_{2m}$.

We now fix $\phi = \phi(Z, \mathcal{M}_1, \ldots, \mathcal{M}_{2m})$, $\bar{\omega} = \bar{\omega}(Z, \mathcal{M}_1, \ldots, \mathcal{M}_{2m})$, so we choose now $\mathcal{M}_1, \ldots, \mathcal{M}_{2m}$ to parametrize our family of small perturbations. As we just have shown, $\mathcal{M}_1, \ldots, \mathcal{M}_{2m}$ can be taken from an arbitrarily large domain in $\mathbb{R}^{2m}$. Let $\mathcal{M}_1, \ldots, \mathcal{M}_{2m}$ be uniformly bounded and let $\mathcal{M}_{2m}$ stay bounded away from zero. As $Z \to +0$, the coefficients $\mathcal{M}_{2m+1}, \ldots, \mathcal{M}_n$ of the characteristic polynomial tend uniformly to zero, according to (3.37). Thus, the characteristic equation

$$\mu^n - \mathcal{M}_1\mu^{n-1} + \cdots + \mathcal{M}_{2m}\mu^{n-2m} - \mathcal{M}_{2m+1}\mu^{n-2m-1} + \cdots + (-1)^n\mathcal{M}_n = 0 \tag{3.64}$$

has $(n-2m)$ roots which tend to zero as $Z \to +0$, and $2m$ roots (we denote them as $\mu_1, \ldots, \mu_{2m}$) which are bounded away from zero and tend to the roots of the polynomial

$$\mu^{2m} - \mathcal{M}_1\mu^{2m-1} + \cdots + \mathcal{M}_{2m}. \tag{3.65}$$

Define the real numbers $\tilde{\mathcal{M}}_1, \ldots, \tilde{\mathcal{M}}_{2m}$ such that $\mu_1, \ldots, \mu_{2m}$ were the roots of the polynomial

$$\mu^{2m} - \tilde{\mathcal{M}}_1\mu^{2m-1} + \cdots + \tilde{\mathcal{M}}_{2m}, \tag{3.66}$$

i.e.

$$\prod_{j=1}^{2m}(\mu - \mu_j) = \mu^{2m} - \tilde{\mathcal{M}}_1\mu^{2m-1} + \cdots + \tilde{\mathcal{M}}_{2m}. \tag{3.67}$$

By construction, $(\tilde{\mathcal{M}}_1, \ldots, \tilde{\mathcal{M}}_{2m})$ tend to $(\mathcal{M}_1, \ldots, \mathcal{M}_{2m})$ as $Z \to +0$. Let us show that $\tilde{\mathcal{M}}_1, \ldots, \tilde{\mathcal{M}}_{2m}$ depend on $\mathcal{M}_1, \ldots, \mathcal{M}_{2m}$ smoothly, and that

$$\frac{d(\tilde{\mathcal{M}}_1, \ldots, \tilde{\mathcal{M}}_{2m})}{d(\mathcal{M}_1, \ldots, \mathcal{M}_{2m})}|_{Z=0} = \mathbb{I}d. \tag{3.68}$$

Indeed, consider the linear operator $\mathcal{Q} : \mathbb{R}^n \to \mathbb{R}^n$ defined by the matrix

$$\begin{pmatrix} 0 & -1 & 0 & \cdots & \cdots \\ 0 & 0 & -1 & 0 & \cdots \\ \vdots & \vdots & \ddots & \ddots & \cdots \\ 0 & \cdots & \cdots & 0 & -1 \\ \mathcal{M}_{2n} & \mathcal{M}_{2n-1} & \cdots & \mathcal{M}_2 & \mathcal{M}_1 \end{pmatrix}. \tag{3.69}$$

Its characteristic equation is also given by (3.64), so it has, as well, $(n-2m)$ eigenvalues close to zero and $2m$ eigenvalues which are bounded away from zero and are the roots of the polynomial (3.66). Hence, the operator $\mathcal{Q}$ has two invariant eigenspaces, one corresponds to the close to zero eigenvalues and the other corresponds to the eigenvalues which are bounded away from zero. The coefficients of the characteristic polynomial of $\mathcal{Q}$ restricted onto the second subspace are exactly the coefficients $\tilde{\mathcal{M}}_1, \ldots, \tilde{\mathcal{M}}_{2m}$. Since all the entries of the matrix

(3.69) are bounded and since it depends smoothly on $\mathcal{M}_1, \ldots, \mathcal{M}_n$, the invariant subspaces depend on $\mathcal{M}_1, \ldots, \mathcal{M}_n$ smoothly as well. This gives us the smooth dependence of $\tilde{\mathcal{M}}_1, \ldots, \tilde{\mathcal{M}}_{2m}$ on $\mathcal{M}_1, \ldots, \mathcal{M}_n$. Recall now that the coefficients $\mathcal{M}_{2m+1}, \ldots, \mathcal{M}_n$ depend on $(\mathcal{M}_1, \ldots, \mathcal{M}_{2m})$ smoothly, and they are continuous in Z along with the derivatives with respect to $(\mathcal{M}_1, \ldots, \mathcal{M}_{2m})$. Thus, the required smooth dependence of $\tilde{\mathcal{M}}_1, \ldots, \tilde{\mathcal{M}}_{2m}$ on $(\mathcal{M}_1, \ldots, \mathcal{M}_{2m})$ at all small Z, including $Z = 0$, follows immediately. Identity (3.68) follows now from the fact that $(\tilde{\mathcal{M}}_1, \ldots, \tilde{\mathcal{M}}_{2m}) = (\mathcal{M}_1, \ldots, \mathcal{M}_{2m})$ at $Z = 0$.

Now, by implicit function theorem, we have that given any values $\tilde{\mathcal{M}}_1, \ldots, \tilde{\mathcal{M}}_{2m}$ with $\tilde{\mathcal{M}}_{2m} \neq 0$, the corresponding values of $\mathcal{M}_1, \ldots, \mathcal{M}_{2m}$ are defined uniquely. In turn, the coefficients $\tilde{\mathcal{M}}_1, \ldots, \tilde{\mathcal{M}}_{2m}$ are defined uniquely by (3.67), given any (symmetric with respect to the complex conjugation) set $\mathcal{R}$ of the non-zero roots $\mu_1, \ldots, \mu_{2m}$. Hence, given any such set $\mathcal{R}$, we find the corresponding values of $\mathcal{M}_1, \ldots, \mathcal{M}_{2m}$, and then the values of the perturbation parameters ϕ and $\bar{\omega}$, for arbitrarily small values of Z. Theorem 3.1 is proven.

By taking in Theorem 3.1 the values of the multipliers $\mu_1, \ldots, \mu_{2m}$ outside the unit circle, we arrive at the following

Corollary 3.1. *Let the assumptions of Theorem 3.1 hold. Then, by an arbitrarily small C^∞-perturbation of system* (3.1)*, a periodic orbit P the instability index $N^+(P)$ of which satisfies*

$$N^+(P) = 2m \tag{3.70}$$

can be born in an arbitrarily small neighborhood of the homoclinic loop under consideration.

Remark 3.2. We note that the unstable manifold $W^u(P)$ of the periodic orbit P constructed in Corollary 3.1 has dimension $2m + 1$. Thus, if every solution of the perturbed system (3.20) can be extended globally for positive $t \in \mathbb{R}_+$, then this unstable manifold is, obviously, a $(2m + 1)$-dimensional invariant submanifold for the system under consideration. Moreover, due to Remark 3.1, we have

$$\dim W^u(P) = [\dim_L(\mathbb{A})], \tag{3.71}$$

where $[v]$ denotes the integral part of v. Since such invariant manifolds always belong to the attractor (if the system possesses a global attractor) then Corollary 3.1 and formula (3.71) present a possibility of obtaining lower bounds for the attractors dimension in terms of their Lyapunov dimension. This possibility will indeed be used in the next Section in order to obtain sharp lower bounds for the attractor's dimension for the abstract hyperbolic equation (1.1).

It is also interesting to consider the case where the multipliers $\mu_1, ..., \mu_{2m}$ in Theorem 3.1 are all equal to 1 in the absolute value. In this case, a small perturbation of the periodic orbit P with the multipliers $\mu_1, ..., \mu_{2m}$ can produce an $(m+1)$-dimensional invariant torus (see a proof in Appendix B). This gives us the following

Corollary 3.2. *Let the assumptions of Theorem 3.1 hold. Then, by an arbitrarily small C^∞-perturbation of system* (3.1)*, an $(m + 1)$-dimensional smooth invariant torus, densely filled by a quasiperiodic trajectory, can be born in an arbitrarily small neighborhood of the homoclinic loop under consideration.*

§4 Lower bounds for the dimension of the attractor.

In this concluding Section, we obtain sharp in the class $\mathbb{S}$ lower bounds for the attractor's dimension for the nonlinear hyperbolic equation (1.1). The main result here is the following theorem.

Theorem 4.1. *For every linear selfadjoint operator* $\mathrm{A} : D(\mathrm{A}) \to \mathrm{H}$ *whose eigenvalues satisfy* (2.1)*, there exist two smooth nonlinear operators* $\mathbb{F}_1$ *and* $\mathbb{F}_2$ *in the form*

$$\mathbb{F}_i(u) := F_i^1((u,e_1),(u,e_2))e_1 + F_i^2((u,e_1),(u,e_2))e_2, \quad u \in \mathrm{H}, \quad i = 1,2, \tag{4.1}$$

where $\{e_i\}_{i=1}^{\infty}$ *is the orthonormal system of eigenvectors associated with* A *and* $F_i^j \in C_0^\infty(\mathbb{R}^2, \mathbb{R})$, $i, j = 1, 2$, *and a smoothing operator* $\Phi = \Phi_{\varepsilon,\gamma,k,m}$ *is defined for every* $\varepsilon > 0$, *every small* $\gamma > 0$ *and every* $k, m \in \mathbb{N}$, *which belongs to the class* $\mathbb{S}$, *see Definition 2.2, and satisfies the estimate*

$$\|\Phi\|_{C_b^k(\mathrm{E}^{-m},\mathrm{H}^m)} \le \varepsilon, \tag{4.2}$$

such that the fractal dimension of the attractor $\mathcal{A} = \mathcal{A}_{\gamma,\varepsilon,k,m}$ *of the equation*

$$\partial_t^2 u + \gamma \partial_t u + \mathrm{A}\, u = \mathbb{F}_1(u) + \gamma \mathbb{F}_2(u) + \Phi(u, \partial_t u) \tag{4.3}$$

possesses the following estimates:

$$C_1 \frac{1}{\gamma} \le \dim_F(\mathcal{A}, \mathrm{E}) \le C_2 \frac{1}{\gamma}, \tag{4.4}$$

where the positive constants C_1 *and* C_2 *are independent of* γ, ε, k *and* m.

Proof. First, take a second order ODE in the form:

$$\partial_t^2 U = U - F_0(U), \quad U \in \mathbb{R}, \tag{4.5}$$

where $F_0 \in C^\infty(\mathbb{R})$ vanishes at the origin together with its first derivative. We assume that equation (4.5) possesses a homoclinic orbit $U_0(t)$ to the equilibrium $U = 0$ (as an example, take $F_0(U) = U^3$). Let us fix a sufficiently small $\gamma > 0$, $n := [1/(2\gamma)] - 1$ and a frequency vector $\omega := (\omega_1, \cdots, \omega_n) \in \mathbb{R}^n$ and consider the following decoupled system of second order ODE's:

$$\begin{cases} \partial_t^2 U(t) = U(t) - F_0(U(t)), \\ \partial_t^2 \bar u_1(t) + \gamma \partial_t \bar u_1(t) + \omega_1^2 \bar u_1(t) = 0, \\ \qquad\qquad \cdots \\ \partial_t^2 \bar u_n(t) + \gamma \partial_t \bar u_n(t) + \omega_n^2 \bar u_n(t) = 0. \end{cases} \tag{4.7}$$

By construction, this system has a homoclinic loop of the type studied in Section 3, so one can expect an analogue of Theorem 3.1 for system (4.7), as it is indeed given by the following lemma.

Lemma 4.1. *Let the above assumptions hold and let, in addition,*

$$\omega_i > \gamma/2, \quad \textit{for every } i = 1, \cdots, n. \tag{4.8}$$

Then, for every $\varepsilon > 0$ *and every* $k \in \mathbb{N}$, *there exist* C_0^∞*-functions* $\Phi_i : \mathbb{R}^{2n+2} \to \mathbb{R}$, $i = 0, 1, \cdots, n$, *satisfying*

$$\|\Phi_i\|_{C_b^k(\mathbb{R}^{2n+2},\mathbb{R})} \le \varepsilon, \tag{4.9}$$

such that the system

$$\begin{cases} \partial_t^2 U(t) = U(t) - F_0(U(t)) + \Phi_0(U(t), \partial_t U(t), \bar{u}(t), \partial_t \bar{u}(t)), \\ \partial_t^2 \bar{u}_1(t) + \gamma \partial_t \bar{u}_1(t) + \omega_1^2 \bar{u}_1(t) = \Phi_1(U(t), \partial_t U(t), \bar{u}(t), \partial_t \bar{u}(t)), \\ \cdots \\ \partial_t^2 \bar{u}_n(t) + \gamma \partial_t \bar{u}_n(t) + \omega_n^2 \bar{u}_n(t) = \Phi_n(U(t), \partial_t U(t), \bar{u}(t), \partial_t \bar{u}(t)) \end{cases} \tag{4.10}$$

possesses a periodic orbit P *with the instability index* $N^+(p) = 2n$.

Proof. We rewrite problem (4.7) in new variables

$$\begin{aligned} &z(t) := (U(t) + \partial_t U(t))/2, \quad w(t) := (U(t) - \partial_t U(t))/2, \\ &\bar{u}_i(t) := \bar{u}_i(t), \qquad \bar{v}_i(t) := (\omega_i^0)^{-1}\left(\partial_t \bar{u}_i(t) + \frac{\gamma}{2}\bar{u}_i(t)\right), \end{aligned} \tag{4.11}$$

where $\omega_i^0 := (\omega_i^2 - \frac{\gamma^2}{4})^{1/2} > 0$, $i = 1, \cdots, n$. In these variables, system (4.7) reads as

$$\begin{cases} \partial_t z = z - \frac{1}{2}F_0(z + w), \\ \partial_t \bar{u}_1 = -\frac{\gamma}{2}\bar{u}_1 + \omega_1^0 \bar{v}_1, \\ \partial_t \bar{v}_1 = -\frac{\gamma}{2}\bar{v}_1 - \omega_1^0 \bar{u}_1, \\ \cdots \\ \partial_t \bar{u}_n = -\frac{\gamma}{2}\bar{u}_n + \omega_n^0 \bar{v}_n, \\ \partial_t \bar{v}_n = -\frac{\gamma}{2}\bar{v}_n - \omega_n^0 \bar{u}_n, \\ \partial_t w = -w + \frac{1}{2}F_0(z + w). \end{cases} \tag{4.12}$$

We now note that system (4.12) has the form of (3.1) and that all assumptions of Theorem 3.1 are, obviously, satisfied for (4.12). Consequently, according to this theorem, for every given $\varepsilon > 0$ and $k \in \mathbb{N}$, there are C_0^∞-functions $\tilde{\Phi}_i : \mathbb{R}^{2n+2} \to \mathbb{R}$, $i = 0, \cdots, 2n+1$, satisfying

$$\|\tilde{\Phi}_i\|_{C_b^k(\mathbb{R}^{2n+2},\mathbb{R})} \le \varepsilon, \tag{4.13}$$

such that the following perturbation of system (4.12)

$$\begin{cases} \partial_t z = z - \frac{1}{2}F_0(z + w) + \tilde{\Phi}_0(z, w, \bar{u}, \bar{v}), \\ \partial_t \bar{u}_i = -\frac{\gamma}{2}\bar{u}_i + \omega_i^0 \bar{v}_i + \tilde{\Phi}_{2i-1}(z, w, \bar{u}, \bar{v}), \\ \partial_t \bar{v}_i = -\frac{\gamma}{2}\bar{v}_i - \omega_i^0 \bar{u}_i + \tilde{\Phi}_{2i}(z, w, \bar{u}, \bar{v}), \quad i = 1, \cdots, n, \\ \partial_t w = -w + \frac{1}{2}F_0(z + w) + \tilde{\Phi}_{2n+1}(z, w, \bar{u}, \bar{v}) \end{cases} \tag{4.14}$$

possesses a periodic orbit P with the instability index $N^+(P) = 2n$. This periodic orbit lies in a small neighborhood of the homoclinic loop of the nonperturbed system, so we may assume without loss of generality that all the functions $\tilde\Phi_i$ have finite supports.

It remains to rewrite system (4.14) as a system of second order ODE's for the variables $U(t) := z(t)+w(t)$ and $\bar u_i(t)$. To this end, we take the sum of the first and the last equation of (4.14) and differentiate it with respect to t and, analogously, we differentiate the equations for $\bar u_i(t)$ in (4.14). This gives us

$$\begin{cases} \partial_t^2 U(t) = U(t) - F_0(U(t)) + \bar\Phi_0(z(t), w(t), \bar u(t), \bar v(t)), \\ \partial_t^2 \bar u_i(t) + \gamma\partial_t \bar u_i(t) + \omega_i^2 \bar u_i(t) = \bar\Phi_i(z(t), w(t), \bar u(t), \bar v(t)), \\ \qquad\qquad i = 1, \cdots, n, \end{cases} \tag{4.15}$$

where the C_0^∞ functions $\bar\Phi_i : \mathbb{R}^{2n+2} \to \mathbb{R}$ satisfy

$$\|\bar\Phi_i\|_{C_b^{k-1}(\mathbb{R}^{2n-2},\mathbb{R})} \le C_k\varepsilon, \tag{4.16}$$

and the constant C_k is independent of ε. In order to finish the proof of the lemma, it remains to express the variables z, w, $\bar v_i$ in terms of U, $\partial_t U$, $\bar u_i$ and $\partial_t \bar u_i$ from the system

$$\begin{cases} z + w = U, \\ z - w = \partial_t U - \tilde\Phi_0(z, w, \bar u, \bar v) - \tilde\Phi_{2n+1}(z, w, \bar u, \bar v), \\ \bar v_i = (\omega_i^0)^{-1}\left(\partial_t \bar u_i + \frac{\gamma}{2}\bar u_i\right) - (\omega_i^0)^{-1}\tilde\Phi_{2i}(z, w, \bar u, \bar v), \\ \qquad\qquad i = 1, \cdots, n. \end{cases} \tag{4.17}$$

Note that in the case when all $\tilde\Phi_i$ are equal to zero identically, system (4.17) reduces to a non-degenerate linear system. Hence, due to (4.13), system of equations (4.17) is indeed can be solved in a unique way (by virtue of the implicit function theorem) if ε is small enough:

$$\begin{cases} z = \Theta_0(U, \partial_t U, \bar u, \partial_t \bar u), \\ \bar v_i = \Theta_i(U, \partial_t U, \bar u, \partial_t \bar u), \quad i = 1, \cdots, n, \\ w = \Theta_{n+1}(U, \partial_t U, \bar u, \partial_t \bar u), \end{cases} \tag{4.18}$$

for some C^∞-functions Θ_i, $i = 0, \cdots, n$. Inserting (4.18) into the right-hand side of (4.15) finishes the proof of Lemma 4.1.

Let us now finish the proof of Theorem 4.1. Indeed, let A be a selfadjoint positive operator in a Hilbert space H with a compact inverse such that its eigenvalues $0 < \lambda_1 \le \lambda_2 \le \cdots$ satisfy condition (2.1) for a certain positive constant κ. Let $\{e_i\}_{i=1}^\infty$ be the corresponding orthonormal system of eigenvectors. Then, every H-valued function $u(t)$, $t \in \mathbb{R}_+$, can be expanded as follows:

$$u(t) := \sum_{i=1}^\infty u_i(t) e_i, \quad u_i(t) := (u(t), e_i). \tag{4.19}$$

Moreover, due to (1.2),

$$\|u(t)\|_{\mathrm{H}^s}^2 := \sum_{i=1}^\infty \lambda_i^s |u_i(t)|^2, \quad s \in \mathbb{R}. \tag{4.20}$$

We rewrite now equation (4.3) in the following equivalent form:

$$\partial_t^2 u_i(t) + \gamma\partial_t u_i(t) + \lambda_i u_i(t) = \bar\Phi_i(u(t), \partial_t u(t)), \quad i = 1, 2, \cdots, \tag{4.21}$$

where $u(t) = (u_1(t), u_2(t), \cdots) \in \mathbb{R}^\infty$ (see (4.19)) and $\{\bar\Phi_i(u, \partial_t u)\}_{i=1}^\infty$ are defined as

$$\bar\Phi_i(u, \partial_t u) := (\mathbb{F}_1(u) + \gamma\mathbb{F}_2(u) + \Phi(u, \partial_t u), e_i). \tag{4.22}$$

We will construct the desired equation (4.3) in the form (4.21). The main idea is to construct the nonlinearities in such a way that the components $u_i(t)$ of the corresponding solution will satisfy system (4.10). Then, by Lemma 4.1, this equation will possess a periodic orbit P such that $N^+(P) = 2n$, $n + 1 := [1/(2\gamma)]$ and, consequently, the fractal dimension of its attractor will be larger than $(2\gamma)^{-1}$. Indeed, let $\gamma > 0$, $\varepsilon > 0$ and let $k \in \mathbb{N}$ be arbitrary. Let us also fix $n := [1/(2\gamma)] - 1$ as in Lemma 4.1. We need to rewrite system (4.10) constructed in Lemma 4.1 in the form (4.21). To this end, we fix the frequencies $\omega_i^2 = \lambda_{i+2}$, $i = 1, \cdots, n$, where λ_i are the eigenvalues of A, and introduce the variables

$$u_1(t) := U(t),\ u_2(t) := \partial_t U(t),\ u_3(t) := \bar u_1(t),\ \cdots,\ u_{n+2}(t) := \bar u_n(t). \tag{4.23}$$

In these variables, (4.10) is written as follows:

$$\begin{cases} \partial_t^2 u_1 + \gamma\partial_t u_1 + \lambda_1 u_1 = \{u_1 - F_0(u_1) + \lambda_1 u_1\} + \\ \qquad + \gamma\{u_2\} + \Phi_1(u_1, \cdots, u_{n+2}, \partial_t u_1, \cdots, \partial_t u_{n+2}), \\ \partial_t^2 u_2 + \gamma\partial_t u_2 + \lambda_2 u_2 = \{u_2 - F_0'(u_1)u_2 + \lambda_2 u_2\} + \\ \qquad + \gamma\{u_1 - F_0(u_1)\} + \Phi_2(u_1, \cdots, u_{n+2}, \partial_t u_1, \cdots, \partial_t u_{n+2}), \\ \partial_t^2 u_3 + \gamma\partial_t u_3 + \lambda_3 u_3 = \\ \qquad = \Phi_3(u_1, \cdots, u_{n+2}, \partial_t u_1, \cdots, \partial_t u_{n+2}), \\ \qquad \cdots \\ \partial_t^2 u_{n+2} + \gamma\partial_t u_{n+2} + \lambda_{n+2} u_{n+2} = \\ \qquad = \Phi_{n+2}(u_1, \cdots, u_{n+2}, \partial_t u_1, \cdots, \partial_t u_{n+2}), \end{cases} \tag{4.24}$$

where the C_0^∞-functions Φ_i satisfy

$$\|\Phi_i\|_{C^{k-2}(\mathbb{R}^{n+2}, \mathbb{R})} \le C_k' \varepsilon. \tag{4.25}$$

Since every solution of (4.10) is, obviously, a solution of (4.24) as well, then (4.24) has also a periodic orbit P with $N^+(P) = 2n$.

We complete system (4.24) as follows:

$$\partial_t^2 u_i + \gamma\partial_t u_i + \lambda_i u_i = 0, \quad i = n+3, n+4, \cdots. \tag{4.26}$$

Then, system (4.24), (4.26) has the form (4.21) indeed. Moreover, since only the existence of a periodic orbit P with $N^+(P) = 2n$ is important for our purposes, we may cut-off all the nonlinearities outside of this orbit and define finally

$$\mathbb{F}_1(u) := \phi_0 \cdot \begin{pmatrix} u_1 - F_0(u_1) + \lambda_1 u_1 \\ u_2 - F_0'(u_1)u_2 + \lambda_2 u_2 \\ 0 \\ 0 \\ \vdots \end{pmatrix}, \quad \mathbb{F}_2(u) := \phi_0 \cdot \begin{pmatrix} u_2 \\ u_1 - F_0(u_1) \\ 0 \\ 0 \\ \vdots \end{pmatrix} \tag{4.27}$$

and

$$
(4.28) \qquad \Phi(u,\partial_t u) := \begin{pmatrix} \Phi_1(u_1,\cdots,u_{n+2},\partial_t u_1,\cdots,\partial_t u_{n+2}) \\ \cdots \\ \Phi_{n+2}(u_1,\cdots,u_{n+2},\partial_t u_1,\cdots,\partial_t u_{n+2}) \\ 0 \\ 0 \\ \vdots \end{pmatrix},
$$

where $\phi_0 := \phi_0(|u_1|^2 + |u_2|^2)$ is an appropriate cut-off function.

Thus, the desired operators $\mathbb{F}_1$, $\mathbb{F}_2$ and Φ are defined. Let us now verify that they satisfy all conditions of Theorem 4.1. Indeed, (4.1) is obvious. Since the operator Φ has a finite rank (see (4.28)), then, obviously, $\Phi \in \mathbb{S}$. Moreover, it follows from (2.1), (4.20) and (4.25) that, for every $m \in \mathbb{N}$

$$
(4.29) \qquad \|\Phi\|_{C^{k-2}(\mathrm{E}^{-m},\mathrm{H}^m)} \le Cn^{4\kappa m}\varepsilon.
$$

Hence, by rescaling, if necessary, ε, we may always satisfy (4.2) (for every fixed m). Furthermore, due to our construction, system (4.24), (4.26) possesses a smooth periodic orbit P with $N^+(P) = 2n$. Therefore,

$$
(4.30) \qquad \dim_F(\mathcal{A},\mathrm{E}) \ge 2n \ge \frac{1}{2\gamma}.
$$

The upper bounds for the fractal dimension in (4.4) is an immediate corollary of Proposition 2.1 and Corollary 2.2. Theorem 4.1 is proven.

Remark 4.1. We emphasize that the operators $\mathbb{F}_1$ and $\mathbb{F}_2$ from Theorem 4.1 have very simple structure (see (4.27)) and can be computed explicitly.

It is also worth to emphasize that the unperturbed system (4.3)

$$
(4.31) \qquad \partial_t^2 u + \gamma\partial_t u + \mathrm{A}\, u = \mathbb{F}_1(u) + \gamma\mathbb{F}_2(u)
$$

possesses a 4-dimensional inertial manifold

$$
M := \{(u_1,u_2,\partial_t u_1,\partial_t u_2) \in \mathbb{R}^4,\ (u_i,\partial_t u_i) = 0,\ i = 3,4,\cdots\}
$$

and, consequently, the fractal dimension of its attractor $\mathcal{A}_0$ satisfies

$$
(4.32) \qquad \dim_F(\mathcal{A}_0,\mathrm{E}) \le 4, \quad \text{for every } \gamma > 0,
$$

whereas its Lyapunov dimension, obviously, satisfies

$$
(4.33) \qquad \dim_L(\mathcal{A}_0,\mathrm{E}) \sim \gamma^{-1}.
$$

Theorem 4.1 shows, however, that, by an arbitrarily small perturbation of equation (4.31), we may increase drastically the fractal dimension of the corresponding attractor $\mathcal{A}$ and obtain the relation

$$
(4.34) \qquad \dim_F(\mathcal{A},\mathrm{E}) \sim \dim_L(\mathcal{A},\mathrm{E}) \sim \gamma^{-1}.
$$

This example confirms that the Lyapunov dimension is a more robust qualitative characteristic of the global attractor than its fractal dimension.

Remark 4.2. We have constructed in Theorem 4.1 the examples of attractors $\mathcal{A}$ of equations of the form (1.1) which depend explicitly on the first derivative $\partial_t u$ of the unknown function u. Differentiating, however, equation (4.3) by t and denoting $v = \partial_t u$ and $w(t) := (u(t), v(t)) \in \tilde{\mathrm{H}} := \mathrm{H} \times \mathrm{H}$, we obtain the equation of the form

$$\partial_t^2 w + \gamma \partial_t w + \tilde{\mathrm{A}} w = \tilde{\mathbb{F}}_1(w) + \gamma \tilde{\mathbb{F}}_2(w) + \tilde{\Phi}(w), \tag{4.35}$$

where the nonlinearities are already independent of $\partial_t w$. Therefore, the phenomena described in Theorem 4.1 can appear in hyperbolic equations of the form (1.1) where the nonlinearity F is independent of $\partial_t u$ ($F(u, \partial_t u) \equiv F(u)$). Unfortunately, this reduction leads to linear operators $\tilde{\mathrm{A}}$ in a special form

$$\tilde{\mathrm{A}} := \begin{pmatrix} \mathrm{A} & 0 \\ 0 & \mathrm{A} \end{pmatrix}. \tag{4.36}$$

In order to avoid this restriction, we permit the explicit dependence of the nonlinearity F on $\partial_t u$ in our abstract model (1.1).

We recall that the usual way of obtaining lower bounds for attractor's fractal dimension is to estimate the instability index for some equilibrium of the equation under consideration, see [BaV89], [Hal87] and references therein (see also [Zel97], where lower estimates for the instability index of a linear nonautonomous equation of type (1.1) with periodic coefficients were given based on the parametrical resonance phenomena). In our next proposition, we show that it is principally impossible, using this method, to obtain reasonable lower bounds for the attractor's dimension of equation (1.1) with nonlinearities belonging to $\mathbb{S}$.

Proposition 4.1. *Let* A *be a linear selfadjoint operator with a compact inverse in a Hilbert space* H *whose eigenvalues satisfy* (2.1) *and let the nonlinearity* F *in equation* (1.1) *belong to the class* $\mathbb{S}$. *Then, for every* $\varepsilon > 0$, *there exists a positive constant* C_ε *such that the instability index of any equilibrium* u_0 *of equation* (1.1) *can be estimated as follows:*

$$N^+(u_0) \le C_\varepsilon \gamma^{-\varepsilon}. \tag{4.37}$$

Proof. Indeed, due to the trick described in Remark 4.2, it is sufficient to prove estimate (4.37) only for the case where

$$F(u, \partial_t u) \equiv F(u). \tag{4.38}$$

Let now u_0 be an arbitrary equilibrium of equation (1.1). Then, the corresponding equation of variations reads

$$\partial_t^2 w + \gamma \partial_t w + \mathrm{A}_{u_0} w = 0, \quad \mathrm{A}_{u_0} := \mathrm{A} - F'(u_0) \tag{4.39}$$

and, consequently, the spectrum of the linearization $D_{u_0} S_t$ at point $(u_0, 0)$ of the semigroup (1.7) can be expressed as follows:

$$\sigma\left(D_{u_0} S_t\right) = \left\{ e^{t\mu_\pm^k}, \ \ \mu_\pm^k := -\frac{\gamma}{2} \pm \left(\frac{\gamma^2}{4} - \theta_k\right)^{1/2} \right\}, \tag{4.40}$$

where $\{\theta_k\}_{k=1}^{\infty} \in \mathbb{C}$ are the eigenvalues of the operator A_{u_0}.

We recall now that the operator F belongs to the class $\mathbb{S}$. Therefore,

$$\|F'(u_0)\|_{\mathcal{L}(\mathrm{H}^{-m},\mathrm{H}^m)} \le C_m, \tag{4.41}$$

for every $m \in \mathbb{N}$, where the constant C_m is independent of $u_0 \in \mathrm{H}$. Thus, due to the classical theory of compact perturbations of selfadjoint operators (see e.g. [GhK69]), we derive from (4.41) that, for every $N \in \mathbb{N}$, there exists a constant C_N such that

$$|\theta_k - \lambda_k| \le C_N k^{-N}, \quad k \in \mathbb{N}, \tag{4.42}$$

where λ_k is the corresponing eigenvalues of the unperturbed operator A. Inserting the asymptotics (4.42) for the eigenvalues θ_k into the formulae (4.40) for $\mu_{\pm}^k$, we obtain after simple computations the desired estimate (4.37). Proposition 4.1 is proved.

Remark 4.3. We stress that our "homoclinic" method of obtaining lower estimates for the attractor's dimension gives, in fact, more than an estimate from below of the maximal attractor. Indeed, in absolutely the same way as Theorem 4.1 was deduced from Corollary 3.1 to Theorem 3.1, we obtain from Corollary 3.2 that the nonlinearities $\mathbb{F}_{1,2}$ and Φ can be constructed in such a way that equation (4.3) would have an invariant torus (densely filled by quasiperiodic trajectories) of dimension $\sim C/\gamma$, where C is a certain constant. In other words, we show that equations under consideration may have minimal sets whose dimension is of the same order as the Lyapunov dimension of the maximal attractor.

Recall also that, according to [NRT78], any quasiperiodic flow on a smooth $(m+1)$-dimensional invariant torus can be perturbed in such a way that the torus would contain an invariant m-dimensional manifold, homeomorphic to $D^{m-1} \times S^1$, the flow on which is smoothly conjugate to a suspension over any aforehand given diffeomorphism of D^{m-1}. Hence, by Corollary 3.2, every dynamics which is possible in a phase space of dimension $\sim C/\gamma$ can be encountered in equation (4.3), at an appropriate choice of the nonlinearities.

Appendix A. Proof of Theorem 1.1 and Lemma 2.1.

In this Appendix, we prove the existence of a solution for problem (1.1) under the assumptions of Theorem 1.1. We also prove the Lipschitz property of the corresponding semigroup S_t, as well as the quasidifferentiability of S_t on the attractor $\mathcal{A}$ (Lemma 2.1). As usual (see [BaV89], [Tem97], [ChV02]), the proof is done via the Galerkin approximation method, based on the a priori estimates (1.6), (1.8), (1.9).

We start with the proof of the a priori estimate (1.6). Let $\xi \in C(\mathbb{R}_+, \mathrm{E})$ be a solution of (1.1). Then, according to assumption (1.4), the nonlinear term $F(u, \partial_t u)$ belongs to the space $C(\mathbb{R}_+, \mathrm{H})$ and is globally bounded in it. Consequently, we may take a scalar product of equation (1.1) with $\partial_t u(t) + \alpha u(t)$, where $\alpha > 0$ is a sufficiently small number, and derive the following relation (see e.g. [Tem97,Lemma II.4.1]):

$$\frac{1}{2}\partial_t[\|\partial_t u(t)\|^2_{\mathrm{H}} + \|u(t)\|^2_{\mathrm{H}^1} + 2\alpha(u(t), \partial_t u(t))] +$$
$$+ (\gamma - \alpha)\|\partial_t u(t)\|^2_{\mathrm{H}} + \alpha\|u(t)\|^2_{\mathrm{H}} = (F, u + \alpha\partial_t u) \le C_\varepsilon + \varepsilon(\|\partial_t u(t)\|^2_{\mathrm{H}} + \|u(t)\|^2_{\mathrm{H}^1}),$$

where $\varepsilon > 0$ can be arbitrarily small. Fixing now $\varepsilon \ll 1$ and $\alpha \ll 1$ in the last inequality and applying the Gronwall's inequality, we obtain (1.6) indeed.

Let us now verify estimate (1.8). Let $\xi_{u_1}(t)$ and $\xi_{u_2}(t)$ be two solutions of (1.1) and let $\eta(t) := [v(t), \partial_t v(t)] := \xi_1(t) - \xi_2(t)$. Then $v(t)$ satisfies the following equation

$$\partial_t^2 v + \gamma\partial_t v + \mathrm{A}\, v = L_1(t)v + L_2(t)\partial_t v, \quad \eta\big|_{t=0} = \xi_1(0) - \xi_2(0),$$

where $L_1(t) := \int_0^1 F'_u(su_1 + (1-s)u_2, s\partial_t u_1)ds$ and $L_2(t) := \int_0^1 F'_{\partial_t u}(su_1 + (1-s)u_2, s\partial_t u_1)ds$. We note that, according to (1.4), these operators are globally bounded as operators from $\mathrm{H}^1 \to \mathrm{H}$ and $\mathrm{H} \to \mathrm{H}$ respectively. Consequently, the right-hand side of the equation for v belongs to the space $C(\mathbb{R}_+, \mathrm{H})$ and we may take a scalar product of this equation with $\partial_t v$ and derive the following estimate

$$\frac{1}{2}[\|\partial_t v(t)\|^2_{\mathrm{H}} + \|v(t)\|^2_{\mathrm{H}^1}] + \gamma\|\partial_t v(t)\|^2_{\mathrm{H}} = \\ = (L_1(t)v(t), \partial_t v(t)) + (L_2(t)\partial_t v(t), \partial_t v(t)) \le K\left(\|v(t)\|^2_{\mathrm{H}^1} + \|\partial_t v(t)\|^2_{\mathrm{H}}\right),$$

where K is independent of u_1 and u_2 (since the derivatives F_u and $F'_{\partial_t u}$ are assumed to be globally bounded). Applying the Gronwall's inequality to this relation, we obtain (1.8).

In order to prove the a priori estimate (1.9), let us differentiate equation (1.1) with respect to t and introduce the function $z(t) := \partial_t u(t)$. Then, this function satisfies the following equation:

$$\text{(A.1)} \qquad \begin{cases} \partial_t^2 z + \gamma\partial_t z + \mathrm{A}\, z = F'_u(u, \partial_t u)z + F'_{\partial_t u}(u, \partial_t u)\partial_t z, \\ z\big|_{t=0} = u'_0, \quad \partial_t z\big|_{t=0} = \partial_t^2 u(0) = -\mathrm{A}\, u_0 - \gamma u'_0 + F(u_0, u'_0). \end{cases}$$

Denote $\eta(t) := [z(t), \partial_t z(t)]$. Since $\xi(0) := [u(0), \partial_t u(0)] \in \mathrm{E}^1$ and the nonlinearity F satisfies (1.4), it follows that $\eta(0) \in \mathrm{E}$ and

$$\text{(A.2)} \qquad \|\eta(0)\|^2_{\mathrm{E}} \le C\left(\|\xi(0)\|^2_{\mathrm{E}^1} + 1\right),$$

for an appropriate constant C depending on F and γ. Moreover, it follows from estimate (1.6) and from equation (1.1) that

$$\text{(A.3)} \qquad \|\eta(t)\|^2_{\mathrm{E}^{-1}} \le C'\|\xi(0)\|^2_{\mathrm{E}}e^{-\gamma t} + C'_1,$$

where C' and C'_1 depend only on A, F and γ.

Taking the inner product in H of equation (A.1) with the function $\partial_t z(t) + \frac{\gamma}{2}z(t)$, we have

$$\text{(A.4)} \quad \partial_t\left\{\|\partial_t z\|^2_{\mathrm{H}} + \|z\|^2_{\mathrm{H}^1} + \gamma\,(z(t), \partial_t z(t))\right\} + \gamma\|\partial_t z\|^2_{\mathrm{H}} + \gamma\|z\|^2_{\mathrm{H}^1} = \\ = 2\,(F'_u(u, \partial_t u)z, \partial_t z) + 2\left(F'_{\partial_t u}(u, \partial_t u)\partial_t z, \partial_t z\right) + \\ + \gamma\,(F'_u(u, \partial_t u)z, z) + \gamma\left(F'_{\partial_t u}(u, \partial_t u)\partial_t z, z\right).$$

Using conditions (1.5), we derive that

$$\text{(A.5)} \quad 2\,(F'_u(u, \partial_t u)z, \partial_t z) + 2\left(F'_{\partial_t u}(u, \partial_t u)\partial_t z, \partial_t z\right) \le \\ \le \frac{\gamma}{2}\left(\|\partial_t z(t)\|^2_{\mathrm{H}} + \|z(t)\|^2_{\mathrm{H}^1}\right) + 4C_\gamma\|[z(t), \partial_t z(t)]\|^2_{\mathrm{E}^{-1}}.$$

Analogously, using assumption (1.4) on the derivatives of F and the interpolation inequality $\|u\|^2_{\mathrm{H}} \le \|u\|_{\mathrm{H}^1}\|u\|_{\mathrm{H}^{-1}}$, we estimate

$$(A.6)\quad \gamma\left(F'_u(u,\partial_t u)z,z\right)+\gamma\left(F'_{\partial_t u}(u,\partial_t u)\partial_t z,z\right)\le \\ \le \frac{\gamma}{8}\left(\|\partial_t z(t)\|^2_{\mathrm{H}}+\|z(t)\|^2_{\mathrm{H}^1}\right)+C'_\gamma\|[z(t),\partial_t z(t)]\|^2_{\mathrm{E}^{-1}}.$$

Inserting estimates (A.5) and (A.6) into the right-hand side of (A.4), we obtain, for a sufficiently small $\gamma>0$,

$$(A.7)\quad \partial_t\left\{\|\partial_t z\|^2_{\mathrm{H}}+\|z\|^2_{\mathrm{H}^1}+\gamma\left(z(t),\partial_t z(t)\right)\right\}+ \\ +\frac{\gamma}{8}\left\{\|\partial_t z\|^2_{\mathrm{H}}+\|z\|^2_{\mathrm{H}^1}+\gamma\left(z(t),\partial_t z(t)\right)\right\}\le C''_\gamma\|\eta(t)\|^2_{\mathrm{E}^{-1}},$$

for an appropriate constant C''_γ which depends only on γ, A and F. Applying the Gronwall's inequality to (A.7) and using (A.2) and (A.3), we arrive at the estimate

$$(A.8)\qquad \|\eta(t)\|^2_{\mathrm{E}}\le C_4 e^{-\gamma t/8}\|\xi(0)\|^2_{\mathrm{E}^1}+C_4.$$

It remains to note that due to equation (1.1) and condition (1.4) we have

$$\|\xi(t)\|^2_{\mathrm{E}^1}\le C\left(\|\eta(t)\|^2_{\mathrm{E}}+1\right),$$

so (1.9) follows from (A.8).

Now we can prove the existence of the solutions of (1.1). Let $\{e_i\}_{i=1}^\infty$ be the orthonormal basis in H generated by the eigenvectors of the selfadjoint operator A and let Π_N be an orthoproector to the first N eigenvectors in H, $\mathrm{H}_N:=\Pi_N\mathrm{H}$ and $\mathrm{E}_N:=\Pi_N\mathrm{E}$. For every $N\in\mathbb{N}$, we consider the following problem in the finite-dimensional space E_N, which approximates the initial infinite-dimensional problem (1.1):

(A.9)

$$\partial_t^2 u_N+\gamma\partial_t u_N+\mathrm{A}\,u_N=\Pi_N F(u_N,\partial_t u_N),\ \ \xi_N(t)\in\mathrm{E}_N,\ \ \xi_N:=[u_N,\partial_t u_N]\big|_{t=0}=\xi_N^0.$$

We recall that $(\Pi_N F,v_N)=(F,v_N)$ for every $v_N\in\mathrm{E}_N$ and, consequently, repeating word by word the derivation of estimates (1.6) and (1.9), we obtain the following uniform (with respect to N) a priori estimates for the solutions of (A.9):

$$(A.10)\qquad \begin{cases} 1.\ \ \|\xi_N(t)\|^2_{\mathrm{E}}\le C\|\xi_N(0)\|_{\mathrm{E}}e^{-\gamma t}+C_1,\\ 2.\ \ \|\xi_N(t)\|^2_{\mathrm{E}^1}\le C_1\|\xi_N(0)\|_{\mathrm{E}^1}e^{-\gamma/8}t+C_3,\end{cases}$$

where the constants C, C_1, C_2 and C_3 are the same as in (1.6) and (1.9) (and, in particular, they are independent of N). On the other hand, equation (A.9) is a system of ODE's with smooth (C^1) nonlinearity and, consequently, estimates (A.10) give the global existence of a solution $\xi_N(t)\in\mathrm{E}_N$ of problem (A.9). Our task now is to pass to the limit $N\to\infty$ in (A.9) and construct the solution $u(t)$ of (1.1) as a limit of Galerkin solutions $u_N(t)$. To this end, we first assume that the initial conditions $\xi(0)$ for problem (1.1) belongs to E^1:

$$(A.11)\qquad \xi(0)\in\mathrm{E}^1\quad\text{and set}\ \ \xi_N^0:=\Pi_N\xi(0).$$

Then, according to (A.10)(2) and equation (A.9), we have the uniform with respect to N estimate

$$\|\partial_t^2 u_N\|_{L^\infty([0,T],\mathrm{H})} + \|\xi_N\|_{L^\infty([0,T],\mathrm{E}^1)} \le C(\|\xi(0)\|_{\mathrm{E}^1}) \tag{A.12}$$

which is valid for every $T > 0$. Consequently, without loss of generality, we may assume that, for every $T > 0$,

(A.13)

$$\xi_N \to \xi \quad *\text{-weakly in } L^\infty([0,T],\mathrm{E}^1) \text{ and } \partial_t^2 u_N \to \partial_t^2 u \quad *\text{-weakly in } L^\infty([0,T],\mathrm{H}),$$

for some function $\xi := [u, \partial_t u] \in L^\infty(\mathbb{R}_+, \mathrm{E}^1)$. We claim that u solves the initial problem (1.1). To this end, we need to pass to the limit $N \to \infty$ in equations (A.9). Indeed, passing to the limit $N \to \infty$ in the linear terms of (A.9) is evident due to (A.13). In order to pass to the limit in the nonlinear term, we recall that the embedding

$$L^\infty([0,T],\mathrm{E}) \cap \{\partial_t^2 u \in L^\infty([0,T],\mathrm{H})\} \subset C([0,T],\mathrm{E})$$

is compact, for every $T > 0$ (see e.g. [Tem97]) and, consequently, (A.13) implies the *strong* convergence $\xi_N \to \xi$ in $C([0,T],\mathrm{E})$. Since the operator F is continuous, it follows that $\Pi_N F(u_N, \partial_t U_N) \to F(u, \partial_t u)$, and u is indeed a solution of problem (1.1).

Thus, for every $\xi(0) \in \mathrm{E}^1$, we have constructed a solution $\xi \in L^\infty(\mathbb{R}_+, E^1) \cap C(\mathbb{R}_+, \mathrm{E})$ of problem (1.1) (moreover, arguing in a standard way like e.g. in [Tem97], one may verify that $\xi \in C(\mathbb{R}_+, \mathrm{E}^1)$, in fact). It is not difficult now to extend this result to the initial data from E. Indeed, let $\xi(0) \in \mathrm{E}$ be an arbitrary initial condition. Let us consider a sequence $\xi^n(0) \in \mathrm{E}^1$ such that

$$\xi^n(0) \to \xi(0) \quad \text{as} \quad n \to \infty \tag{A.14}$$

Let also $\xi^n(t) \in \mathrm{E}^1$ be the corresponding solutions of problem (1.1), the existence of which has just been proven. Then, according to estimate (1.8), there exists some function $\xi \in C(\mathbb{R}_+, \mathrm{E})$ such that

$$\xi^n \to \xi \quad \text{in} \quad C([0,T],\mathrm{E}) \quad \text{strongly,} \tag{A.15}$$

for every $T > 0$. As above, the *strong* convergence law (A.15) allows to pass to the limit $n \to \infty$ in the equations for u^n and verify that the limit function $u(t)$ also satisfies equation (1.1). Thus, the existence of a solution of problem (1.1) is now proven under the assumptions of Theorem 1.1.

It remains to prove the quasidifferentiability of the corresponding semigroup S_t on the attractor $\mathcal{A}$. Let $\xi_1(t)$ and $\xi_2(t)$ be two solutions of problem (1.1) belonging to $\mathcal{A}$ and let $v(t)$ be a solution of equation of variations (2.7) (computed along the trajectory $\xi_1(t)$) with $[v(0), \partial_t v(0)] = \xi_1(0) - \xi_2(0)$. Then, arguing as in the derivation of estimate (1.8), we obtain the following estimate:

$$\|[v(t), \partial_t v(t)]\|_{\mathrm{E}}^2 \le C\|[v(0), \partial_t v(0)]\|_{\mathrm{E}}^2 e^{Kt}, \tag{A.16}$$

where the constants C and K are independent of u_1 and u_2. Moreover, the function $w(t) := u_1(t) - u_2(t) - v(t)$ obviously satisfies the following linear nonhomogeneous equation

(A.17)

$$\partial_t^2 w + \gamma \partial_t w + \mathrm{A}\, w - F'_u(u_1, \partial_t u_1) w - F'_{\partial_t u}(u_1, \partial_t u_1) \partial_t w = h_{u_1,u_2}(t), \ [w, \partial_t w]\big|_{t=0} = 0,$$

where

$$\text{(A.18)}\quad h_{u_1,u_2}(t) := \left\{\int_0^1 (F'_u(su_1 + (1-s)u_2, s\partial_t u_1) - F'_u(u_1, \partial_t u_1)\, ds\right\} v(t)+$$
$$+\left\{\int_0^1 (F'_{\partial_t u}(su_1 + (1-s)u_2, s\partial_t u_1) - F'_u(u_1, \partial_t u_1)\, ds\right\} \partial_t v(t).$$

Consequently, analogously to (1.8) and (A.16), we have

$$\text{(A.19)}\qquad \|[w(T), \partial_t w(T)]\|_{\mathrm{E}} \le Ce^{KT}\int_0^T \|h_{u_1,u_2}(t)\|^2_{\mathrm{H}}\, dt,$$

where the constants C and K are independent of u_1 and u_2. On the other hand, since the attractor $\mathcal{A}$ is compact in E, the set

$$\mathcal{A}_{1,2} := \{s\xi_1 + (1-s)\xi_2,\quad \xi_1, \xi_2 \in \mathcal{A},\quad s \in [0,1]\}$$

is also compact in E and, due to assumption (1.4), the derivatives F'_u and $F'_{\partial_t u}$ are uniformly continuous on this set. Consequently, we have an estimate

$$\text{(A.20)}\qquad \|h_{u_1,u_2}(t)\|_{\mathrm{H}} \le \alpha(\|\xi_1(t) - \xi_2(t)\|_{\mathrm{E}})\|[v(t), \partial_t v(t)]\|_{\mathrm{E}},$$

where the function $\alpha(z)$ tends to zero as $z \to 0^+$ and is independent of t, u_1 and u_2. Estimates (1.8), (A.16), (A.19) and (A.20) imply that

$$\text{(A.21)}\qquad \|[w(T), \partial_t w(T)]\|^2_{\mathrm{E}} \le \alpha_T(\|\xi_1(0) - \xi_2(0)\|_{\mathrm{E}})\|\xi_1(0) - \xi_2(0)\|_{\mathrm{E}},$$

where the function $\alpha_T(z)$ tends to zero as $z \to 0^+$, it depends on $T \ge 0$ but it is independent of $\xi_1, \xi_2 \in \mathcal{A}$. Thus, estimate (2.2) is verified. The continuity of $S'(\xi)$ on the attractor is verified analogously. End of the proof.

Appendix B. Proof of Corollary 3.2.

Let P be a periodic orbit, born by a small perturbation of a homoclinic loop of system (3.1), which has $2m$ multipliers equal to 1 in the absolute value (such an orbit can indeed be born according to Theorem 3.1). Let us prove that a smooth $(m+1)$-dimensional invariant torus, filled by quasiperiodic trajectories each of which is dense in the torus, can be born at the bifurcations of P.

Proof. Consider a Poincar'e map $(x \mapsto \bar{x})$ for the periodic orbit P:

$$\text{(B.1)}\qquad \bar{x} = Bx + o(x),$$

here $x = 0$ is the fixed point which corresponds to the orbit P. The eigenvalues of the matrix B are the multipliers of P. By assumption, $2m$ of these eigenvalues lie on the unit circle. Our goal is to prove that this map can be perturbed in such a way that an m-dimensional invariant torus would appear in a small neighborhood of the origin.

Fix any integer $r \ge 1$. It is obvious, that by perturbations, small in C^r-topology, one can arrange arbitrary small changes in any entries of the matrix B in (B.1).

Hence, we can always achieve that B would have exactly m pairs of complex-conjugate eigenvalues on the unit circle: $e^{\pm i\omega_1}, \dots, e^{\pm i\omega_m}$, all ω's are rationally independent and the factors ω_j/π are irrational; and the rest of the multipliers does not lie on the unit circle.

By the center manifold theorem, our map has a $2m$-dimensional smooth invariant manifold which is tangent at $x = 0$ to the eigenspace of B corresponding to the multipliers on the unit circle. It is well-known that since the numbers $\{\pi, \omega_1, \dots, \omega_m\}$ are rationally independent, there exist local coordinates $(z_1, \dots, z_m) \in \mathbb{C}^m$ in which the map on the center manifold takes the following (normal) form

$$\bar{z}_j = Q_j(z_1 z_1^*, \dots, z_m z_m^*) z + o(z^r) \qquad (j = 1, \dots, m) \tag{B.2}$$

where * means complex conjugation, and Q_j are complex polynomials of degree $\leq (r-1)/2$,

$$Q_j(0) = e^{i\omega_j}. \tag{B.3}$$

It is obvious that by an arbitrary small (in the C^r-topology) perturbation, one can make map (B.2) coincide with the polynomial map

$$\bar{z}_j = Q_j(z_1 z_1^*, \dots, z_m z_m^*) z \qquad (j = 1, \dots, m) \tag{B.4}$$

in a sufficiently small neighborhood of zero. Thus, it is enough to prove that a small perturbation of map (B.4) can produce an m-dimensional invariant torus arbitrarily close to $z = 0$.

Let us, first, introduce polar coordinates $\rho_j e^{i\varphi_j} := z_j$, $j = 1, \dots, m$. Map (B.4) recasts as

$$\begin{cases} \bar{\rho}_j = R_j(\rho_1^2, \dots, \rho_m^2)\rho_j \\ \bar{\varphi}_j = \varphi_j + \Omega_j(\rho_1^2, \dots, \rho_m^2) \end{cases} \qquad (j = 1, \dots, m) \tag{B.5}$$

where $Q_j \equiv R_j e^{i\Omega_j}$ $(j = 1, \dots, m)$, so R_j, Ω_j are real analytic functions, and

$$R_j(0) = 1, \quad \Omega_j(0) = \omega_j \qquad (j = 1, \dots, m). \tag{B.6}$$

Let $a_1, \dots, a_m, \theta_1, \dots, \theta_m$ be arbitrary small numbers. Then the map

$$\begin{cases} \bar{\rho}_j = (a_j + R_j(\rho_1^2, \dots, \rho_m^2))\rho_j \\ \bar{\varphi}_j = \varphi_j + \theta_j + \Omega_j(\rho_1^2, \dots, \rho_m^2) \end{cases} \qquad (j = 1, \dots, m) \tag{B.7}$$

is a small (real analytic) perturbation of (B.5). The amplitude map

$$\bar{\rho}_j = (a_j + R_j(\rho_1^2, \dots, \rho_m^2))\rho_j \qquad (j = 1, \dots, m) \tag{B.8}$$

is independent here on the phases $\varphi_1, \dots, \varphi_m$. Therefore, a fixed point of (B.8) with all $\rho_1, \dots, \rho_m$ non-zero corresponds to an m-dimensional invariant torus of (B.7).

Take now some sufficiently small strictly positive numbers $\rho_1^0, \ldots, \rho_m^0$. Put

$$\text{(B.9)} \qquad \begin{cases} a_j = 1 - R_j((\rho_1^0)^2, \ldots, (\rho_m^0)^2), \\ & (j = 1, \ldots, m) \\ \theta_j = \omega_j - \Omega_j((\rho_1^0)^2, \ldots, (\rho_m^0)^2). \end{cases}$$

By (B.6), the numbers $a_1, \ldots, a_m, \theta_1, \ldots, \theta_m$ are indeed small when $\rho_1^0, \ldots, \rho_m^0$ are small. With $a_1, \ldots, a_m$ given by (B.9), $(\rho_1^0, \ldots, \rho_m^0)$ is a fixed point of (B.8), i.e. the torus $(\rho_1 = \rho_1^0, \ldots, \rho_m = \rho_m^0)$ is a smooth m-dimensional invariant torus of (B.7). By construction, map (B.7) is a C^r-small perturbation of the restriction of the original Poincaré map on the center manifold, so we have indeed constructed the perturbation under which an invariant torus is born from our periodic orbit. By (B.6), (B.7), the restriction of the Poincaré map on this torus is given by

$$\bar{\varphi}_j = \varphi_j + \omega_j \qquad (j = 1, \ldots, m).$$

Since the numbers $\pi, \omega_1, \ldots, \omega_m$ are rationally independent, it follows that every orbit of this map is indeed quasiperiodic and fills the torus densely. End of the proof.

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