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GEVREY REGULARITY FOR THE ATTRACTOR OF A DAMPED WAVE EQUATION

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Abstract. The goal of this paper is to obtain time-asymptotic regularity in Gevrey spaces of the solution of a damped wave equation. The difficulty is due to the fact that this equation is only partially dissipative.

1. **Introduction.** We consider the following singularly perturbed damped wave equation in a cube domain $\Omega = [0, 2\pi]^3$

$$\begin{aligned} &\varepsilon\partial_t^2 u^\varepsilon + \gamma\partial_t u^\varepsilon + Au^\varepsilon + f(u^\varepsilon) = g, \\ &u^\varepsilon_{|t=0} = u_0,\ \partial_t u^\varepsilon_{|t=0} = u_1, \end{aligned} \tag{1}$$

where the operator $A = I - \triangle$ with periodic boundary conditions.
We assume that $\varepsilon > 0$ and $\gamma > 0$. The nonlinear function f is required to be real analytic,

$$f(u) = \sum_{j=0}^{\infty} a_j u^j, \text{ where } h(s) = \sum_{j=0}^{\infty} |a_j| s^j < +\infty\ \forall s \in \mathbb{R}. \tag{2}$$

We assume furthermore that the nonlinearity f satisfies

$$\begin{aligned} &f'(u) \geq -K, \\ &f(u)\cdot u \geq 0 \text{ if } |u| \geq L \\ &|f''(u)| \leq C(1 + |u|), \end{aligned} \tag{3}$$

where C, K, and L are fixed positive constants. The assumptions (2) and (3) are fulfilled for cubic nonlinearity $f(u) = u^3 - \alpha u,\ \alpha \in \mathbb{R}$.

Remark 1. We can replace the assumption (3) by an other one, if we are able to obtain uniform (with respect to ε) absorbing sets in $L^\infty(\Omega)$. For example $f(u) = \sin u$.

We assume that

$$g \text{ is periodic and analytic.} \tag{4}$$

In [1] [2], we obtained, for this problem, the existence of exponential attractors with a rate of attraction, a diameter and a fractal dimension uniform with ε, in the

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variable u as well as u_t. For such a result, the appropriated spaces for solutions of (1) are $\mathcal{E}^k(\varepsilon) = H^{k+1}_{per}(\Omega) \times H^k_{per}(\Omega)$ equipped with the following norms

$$||(u^\varepsilon(t), u^\varepsilon_t(t))||^2_{\mathcal{E}^k(\varepsilon)} = \varepsilon||u^\varepsilon_t(t)||^2_{H^k_{per}} + ||u^\varepsilon_t(t)||^2_{H^{k-1}_{per}} + ||u^\varepsilon(t)||^2_{H^{k+1}_{per}},$$

the spaces $H^k_{per}(\Omega)$ denote the classical Sobolev spaces on Ω with periodic boundary conditions.
We are able to prove the existence of smooth exponential attractors in $\mathcal{E}^k(\varepsilon)$ for k large, attracting all sets of $\mathcal{E}^0(\varepsilon)$ even if the equation is not fully dissipative, thanks to a transitivity property [2]. Here we show a stronger result of regularity with Gevrey classes of the assymptotic trajectories. We state without detailed proof the existence of exponential attractors with Gevrey regularity attracting all sets of $\mathcal{E}^0(\varepsilon)$.
Gevrey regularity for solution of dissipative partial dissipation equation is obtained for example for Navier-Stokes equations in [4]. More recently, Gevrey regularity for asymptotic trajectories of partially dissipative problems is obtained in [5] for a Bénard Convection model.

2. **Main result.** We introduce the Gevrey classes $G^p_\sigma(\Omega) = D(A^{\frac{p}{2}} e^{\sigma A^{\frac{1}{2}}})$. The norm on $G^p_\sigma(\Omega)$ is

$$||u||^2_{G^p_\sigma(\Omega)} = \sum_{j\in\mathbb{Z}^3} |u_j|^2(1+j^2)^{\frac{p}{4}} e^{2\sigma(1+j^2)^{\frac{1}{2}}},$$

where the u_j are the Fourier coefficients of u.
Let us introduce the Gevrey classes $\mathcal{F}^k_\sigma(\varepsilon) = G^{k+1}_\sigma(\Omega) \times G^k_\sigma(\Omega)$ for our problem, equipped with the norm

$$||(u^\varepsilon(t), u^\varepsilon_t(t))||^2_{\mathcal{F}^k_\sigma(\varepsilon)} = \varepsilon||u^\varepsilon_t(t)||^2_{G^k_\sigma} + ||u^\varepsilon_t(t)||^2_{G^{k-1}_\sigma} + ||u^\varepsilon(t)||^2_{G^{k+1}_\sigma}.$$

We denote by $S^\varepsilon(t)$ the semigroup associated to (1),

$$S^\varepsilon(t)(u_0, u_1) = (u^\varepsilon(t), u^\varepsilon_t(t)).$$

The aim of this paper is to establish

Theorem 1. *Let $k > \frac{5}{2}$, under assumptions (2), (3), (4), for all (u_0, u_1) in $\mathcal{B} \subset \mathcal{E}^0(\varepsilon)$, there exist $\varepsilon_{\max}$ and $\sigma_{\max}$ such that for $\varepsilon \leq \varepsilon_{\max}$ and $\sigma \leq \sigma_{\max}$, , there exist (v, v_t) uniformly bounded with respect to ε in $L^\infty(\mathbb{R}^+, \mathcal{F}^k_\sigma(\varepsilon))$ such that*

$$||(u^\varepsilon(t), u^\varepsilon_t(t)) - (v(t), v_t(t))||_{\mathcal{E}^0(\varepsilon)} \leq C \exp(-\mu t), \ \forall t \geq 0,$$

with $\mu > 0$ and C independent of ε.

Remark 2. Because of the lack of time regularizing effect in the wave equation, we can't obtain an estimate in $\mathcal{F}^k_t(\varepsilon)$) for $(u^\varepsilon(t), u^\varepsilon_t(t))$ as it is made for dissipative equations. But we obtain a time-asymptotic regularizing effect.

Corollary 1. *Under the same assumptions, the points of the attractor of (1) are a uniformly bounded for the $\mathcal{F}^k_\sigma(\varepsilon)$-norm.*

Theorem 2. *Under the same assumption than theorem 1, there exist exponential attractors $\mathcal{M}^\varepsilon \subset \mathcal{F}^k_\sigma(\varepsilon)$ on $\mathcal{E}^0(\varepsilon)$. The radius of $\mathcal{M}^\varepsilon$ is uniformly bounded on $\mathcal{F}^k_\sigma(\varepsilon)$ with respect to $\varepsilon \leq \varepsilon_{\max}$. The rate of attraction is also uniform, the fractal dimension has a uniform bound.*

We don't prove this theorem in this article. The first step of the proof is to show that trajectories stemmed from a bounded set of $\mathcal{F}^k_\sigma(\varepsilon)$ are still bounded for all time in $\mathcal{F}^k_\sigma(\varepsilon)$. the second step is to use a transitivity argument as in [2] or as in the end of the proof of the Theorem 1.

3. **Proof of the Theorem 1.** The proof of this theorem uses techniques developed in [5] and [4] and uses results of [2].

Lemma 1. [4] *Let u and v be in $G^k_\sigma(\Omega)$, if $k > \frac{n}{2}$ then uv belongs to $G^k_\sigma(\Omega)$ and there exists C_k such that,*

$$||uv||_{G^k_\sigma(\Omega)} \leq C_k ||u||_{G^k_\sigma(\Omega)} ||v||_{G^k_\sigma(\Omega)}.$$

Let f be a function verifying assumption (2) with a majorizing function h, then,

$$||f(u)||_{G^k_\sigma(\Omega)} \leq (1 + C_k^{-1}) h(C_k ||u||_{G^k_\sigma(\Omega)}). \tag{5}$$

Let λ be an eigenvalue of A, let P_λ be the projector on the low frequencies (the subspace generated by the eigenfunctions whose eigenvalues are smaller than λ). We construct (v, v_t) in the following way

$$(v, v_t) = P_\lambda(u^\varepsilon, u^\varepsilon_t) + (\hat{v}, \hat{v}_t),$$

where $(\hat{v}, \hat{v}_t)$ is solution of

$$\begin{array}{l} \varepsilon \hat{v}_{tt} + \alpha \hat{v}_t - \triangle \hat{v} + Q_\lambda f(P_\lambda u + \hat{v}) = Q_\lambda g, \\ \hat{v}_{|t=0} = 0, \hat{v}_{t|t=0} = 0, \\ \text{with } Q_\lambda = I - P_\lambda. \end{array} \tag{6}$$

In order to prove this theorem 1, we first assume that (u_0, u_1) belongs to $\mathcal{E}^k_\sigma(\varepsilon)$.

Lemma 2. *The solution (v, v_t) of (6) belongs to $L^\infty(\mathbb{R}^+, \mathcal{F}^k_\sigma(\varepsilon))$*

Proof of Lemma. In order to obtain an estimate in $L^\infty(\mathbb{R}^+, \mathcal{F}^k_\sigma(\varepsilon))$, we compute the $\mathcal{G}^k_\sigma(\varepsilon)$ inner product of (6) with $\hat{v}$, $\hat{v}_t$ and $A^{-1}\hat{v}_{tt}$. We combine these three equations and denote $B = A^{\frac{1}{2}}$, we obtain,

$$\begin{array}{c} \frac{d}{dt}\left(\frac{\varepsilon}{2}|B^k e^{\sigma B}\hat{v}_t|_2^2 + \frac{1}{2}|B^{k+1}e^{\sigma B}\hat{v}|_2^2 + \alpha\varepsilon(B^k e^{\sigma B}\hat{v}_t, B^k e^{\sigma B}\hat{v}) + \frac{\alpha}{2}|B^k e^{\sigma B}\hat{v}|_2^2\right) \\ + \frac{d}{dt}\left(\frac{\beta}{2}|B^{k-1}e^{\sigma B}\hat{v}_t|_2^2 + \beta(B^k e^{\sigma B} v, B^k e^{\sigma B} v_t)\right) \\ + \frac{d}{dt}\left(\beta(B^{k-1}e^{\sigma B}Q_\lambda(f(P_\lambda u^\varepsilon + \hat{v})), B^{k-1}e^{\sigma B}\hat{v}_t) - \beta(B^{k-1}e^{\sigma B}g, B^{k-1}e^{\sigma B}\hat{v}_t)\right) \\ +(\gamma - \alpha\varepsilon - \beta)|B^k e^{\sigma B}\hat{v}_t|_2^2 + \alpha|B^{k+1}e^{\sigma B}\hat{v}|_2^2 + \beta\varepsilon|B^{k-1}e^{\sigma B}\hat{v}_{tt}|_2^2 \leq \\ \beta(B^k e^{\sigma B}\hat{v}, B^k e^{\sigma B}\hat{v}_t) + \beta(B^{k-1}e^{\sigma B}Q_\lambda f'(P_\lambda u^\varepsilon + \hat{v})\partial_t(P_\lambda u^\varepsilon + \hat{v}), B^{k-1}e^{\sigma B}\hat{v}_t) \\ -\alpha(B^k e^{\sigma B}Q_\lambda f(P_\lambda u^\varepsilon + \hat{v}), B^k e^{\sigma B}\hat{v}) + \alpha(B^k e^{\sigma B}g, B^k e^{\sigma B}\hat{v}) \\ +(B^k e^{\sigma B}g, B^k e^{\sigma B}\hat{v}_t) - (B^k e^{\sigma B}Q_\lambda f(P_\lambda u^\varepsilon + \hat{v}), B^k e^{\sigma B}\hat{v}_t). \end{array} \tag{7}$$

The constant α and β will be chosen properly and independent of ε. A first step is to take $\beta = 0$, we then obtain the estimate,

$$\begin{array}{r} \frac{d}{dt}\Gamma + (\gamma - \alpha\varepsilon)||\hat{v}_t||^2_{G^k_\sigma} + \alpha||\hat{v}||^2_{G^{k+1}_\sigma} \leq \alpha||Q_\lambda f(P_\lambda u^\varepsilon + \hat{v})||_{G^k_\sigma}||\hat{v}||_{G^k_\sigma} \\ +\alpha||g||_{G^k_\sigma}||\hat{v}||_{G^k_\sigma} + ||Q_\lambda f(P_\lambda u^\varepsilon + \hat{v})||_{G^k_\sigma}||\hat{v}_t||_{G^k_\sigma} \\ +||g||_{G^k_\sigma}||\hat{v}_t||_{G^k_\sigma} \end{array} \tag{8}$$

with

$$\Gamma = \frac{\varepsilon}{2}||\hat{v}_t||^2_{G^k_\sigma} + \frac{1}{2}||\hat{v}||^2_{G^{k+1}_\sigma} + \frac{\alpha}{2}||\hat{v}||^2_{G^k_\sigma} + \alpha\varepsilon(\hat{v}_t, \hat{v})_{G^k_\sigma}.$$

According to Cauchy-Schwartz inequality, and arguing that λ is the smallest eigenvalue on $Q_\lambda L^2(\Omega)$, we obtain,

$$\frac{d}{dt}\Gamma + (\frac{\gamma}{2} - \alpha\varepsilon)||\hat{v}_t||^2_{G^k_\sigma} + \alpha||\hat{v}||^2_{G^{k+1}_\sigma} \leq (\frac{\alpha^2}{\lambda} + \frac{1}{\gamma})(||f(P_\lambda u^\varepsilon + \hat{v})||^2_{G^k_\sigma} + ||g||^2_{G^k_\sigma}) \quad (9)$$

We choose $\varepsilon_{\max} \leq \frac{\gamma}{4\alpha}$, then,

$$\frac{\varepsilon}{4}||\hat{v}_t||^2_{G^k_\sigma} + \frac{1}{2}||\hat{v}||^2_{G^{k+1}_\sigma} + \frac{\alpha}{4}||\hat{v}||^2_{G^k_\sigma} \leq \Gamma \leq \frac{3\varepsilon}{4}||\hat{v}_t||^2_{G^k_\sigma} + \frac{1}{2}||\hat{v}||^2_{G^{k+1}_\sigma} + \frac{3\alpha}{4}||\hat{v}||^2_{G^k_\sigma}. \quad (10)$$

Choosing $\alpha \leq 4\lambda$, we have

$$\frac{d}{dt}\Gamma + \alpha\Gamma \leq (\frac{\alpha^2}{\lambda} + \frac{1}{\gamma})(||f(P_\lambda u^\varepsilon + \hat{v})||^2_{G^k_\sigma} + ||g||^2_{G^k_\sigma}). \quad (11)$$

According to assumption (2) and applying (5), we have,

$$\frac{d}{dt}\Gamma + \alpha\Gamma \leq (\frac{\alpha^2}{\lambda} + \frac{1}{\gamma})((1 + C_k^{-1})h^2(C_k||P_\lambda u^\varepsilon + \hat{v}||_{G^k_\sigma}) + ||g||^2_{G^k_\sigma}). \quad (12)$$

Lemma 3. *Assume that $\sigma \leq \sigma_{\max} \leq \frac{c}{\lambda}$, then,*

$$||P_\lambda u^\varepsilon||^2_{G^k_\sigma} \leq e^{\sigma_{\max}\lambda}||u^\varepsilon||^2_{H^k_{per}} \leq e^c||u^\varepsilon||^2_{H^k_{per}} \leq C,$$

where c is a fixed constant independent of λ.

Proof of Lemma. The first inequality of the lemma is obvious, the second follows from the existence of absorbing sets [2] in $\mathcal{E}^k_\sigma$ assuming the same regularity on the initial data. □

We can now conclude to the bound of Γ instead of the *a priori* high growth of h by virtue of the smallness of initial data. As a matter of fact, $\Gamma(t = 0) = 0$, then, while $\Gamma \leq m$, we have,

$$||\hat{v}||^2_{G^k_\sigma} \leq \frac{4m}{\alpha}$$

$$\frac{d}{dt}\Gamma + \alpha\Gamma \leq (\frac{\alpha^2}{\lambda} + \frac{1}{\gamma})((1 + C_k^{-1})h^2(C_k(\sqrt{C} + 2\sqrt{\frac{m}{\alpha}})) + ||g||^2_{G^k_\sigma})$$

We deduce that, during the time such that $\Gamma \leq m$

$$\Gamma \leq (\frac{\alpha}{\lambda} + \frac{1}{\alpha\gamma})((1 + C_k^{-1})h^2(C_k(\sqrt{C} + 2\sqrt{\frac{m}{\alpha}})) + ||g||^2_{G^k_\sigma})$$

Choosing α and λ ($\lambda \geq \frac{\alpha}{4}$) large enough so that the right handside is smaller than m, we have shown that Γ remains smaller than m for all time under the condition

$$m > (\frac{\alpha}{\lambda} + \frac{1}{\alpha\gamma})((1 + C_k^{-1})h^2(C_k\sqrt{C}) + ||g||^2_{G^k_\sigma}).$$

For example we take

$$\alpha = \lambda \geq \max(\frac{1}{\gamma}, \frac{4m}{C^2}), \text{ with } m = 2((1 + C_k^{-1})h^2(2C_k\sqrt{C}) + ||g||^2_{G^k_\sigma}).$$

¿From the inequality (10), we then have obtained a bound for $\varepsilon||\hat{v}_t||^2_{G^k_\sigma} + 2||\hat{v}||^2_{G^{k+1}_\sigma} + \alpha||\hat{v}||^2_{G^k_\sigma}$ assuming that $(\tilde{u}_0, \tilde{u}_1)$ belongs to $\mathcal{E}^k$. We go back to (7) with the same value $\alpha(=\lambda)$, $\varepsilon_{\max}$ as below but with $\beta \neq 0$. The goal is now to obtain a bound on $\hat{v}_t$ in G_σ^{k-1} independently of ε.

$$\frac{d}{dt}\left(\Gamma + \frac{\beta}{2}|B^{k-1}e^{\sigma B}\hat{v}_t|_2^2 + \beta(B^k e^{\sigma B} v, B^k e^{\sigma B} v_t)\right)$$
$$+\frac{d}{dt}\left(\beta(B^{k-1}e^{\sigma B}Q_\lambda(f(P_\lambda u^\varepsilon + \hat{v})), B^{k-1}e^{\sigma B}\hat{v}_t) - \beta(B^{k-1}e^{\sigma B}g, B^{k-1}e^{\sigma B}\hat{v}_t)\right)$$
$$+\frac{\gamma}{16}|B^k e^{\sigma B}\hat{v}_t|_2^2 + \alpha\Gamma \leq$$
$$\beta(B^k e^{\sigma B}\hat{v}, B^k e^{\sigma B}\hat{v}_t) + \beta(B^{k-1}e^{\sigma B}Q_\lambda f'(P_\lambda u^\varepsilon + \hat{v})\partial_t(P_\lambda u^\varepsilon + \hat{v}), B^{k-1}e^{\sigma B}\hat{v}_t) + \alpha m.$$

We denote by Γ_1 the quantity,

$$\Gamma_1 = \beta(B^k e^{\sigma B} v, B^k e^{\sigma B} v_t) + \beta(B^{k-1}e^{\sigma B}Q_\lambda(f(P_\lambda u^\varepsilon + \hat{v})), B^{k-1}e^{\sigma B}\hat{v}_t)$$
$$-\beta(B^{k-1}e^{\sigma B}g, B^{k-1}e^{\sigma B}\hat{v}_t).$$

Then,

$$\frac{d}{dt}\left(\Gamma + \frac{\beta}{2}|B^{k-1}e^{\sigma B}\hat{v}_t|_2^2 + \Gamma_1\right) + \frac{\gamma}{16}|B^k e^{\sigma B}\hat{v}_t|_2^2 + \alpha(\Gamma + \Gamma_1) \leq$$
$$\beta(B^k e^{\sigma B}\hat{v}, B^k e^{\sigma B}\hat{v}_t) + \beta(B^{k-1}e^{\sigma B}Q_\lambda f'(P_\lambda u^\varepsilon + \hat{v})\partial_t(P_\lambda u^\varepsilon + \hat{v}), B^{k-1}e^{\sigma B}\hat{v}_t)$$
$$+\alpha(\Gamma_1 + m).$$

We can bound Γ_1 in the following way,

$$\alpha\Gamma_1 \leq \frac{\gamma}{32}||\hat{v}_t||_{G_\sigma^k}^2 + 32\lambda\beta\gamma^{-1}m + 16\beta^2\gamma^{-1}(h(\sqrt{C} + \sqrt{\frac{m}{\lambda}}) + ||g||_{G_\sigma^k})^2.$$

Furthermore,

$$\beta(B^k e^{\sigma B}\hat{v}, B^k e^{\sigma B}\hat{v}_t) + \beta(B^{k-1}e^{\sigma B}Q_\lambda f'(P_\lambda u^\varepsilon + \hat{v})\partial_t(P_\lambda u^\varepsilon + \hat{v}), B^{k-1}e^{\sigma B}\hat{v}_t) \leq$$
$$\frac{\gamma}{64}||\hat{v}_t||_{G_\sigma^k}^2 + 64\beta^2\gamma^{-1}\lambda^{-1}m + 64\gamma^{-1}\lambda^{-\frac{3}{2}}\sqrt{C}h'(\sqrt{C} + \sqrt{\tfrac{m}{\lambda}})$$
$$+2048\gamma^{-2}\lambda^{-4}h'^2(\sqrt{C} + \sqrt{\tfrac{m}{\lambda}}).$$

Choosing $\beta = \frac{\gamma}{32}$, we obtain,

$$\frac{d}{dt}\left(\Gamma + \frac{\beta}{2}|B^{k-1}e^{\sigma B}\hat{v}_t|_2^2\right) + \lambda\left(\Gamma + \frac{\beta}{2}|B^{k-1}e^{\sigma B}\hat{v}_t|_2^2\right) \leq$$
$$64\beta^2\gamma^{-1}\lambda^{-1}m + 64\gamma^{-1}\lambda^{-\frac{3}{2}}\sqrt{C}h'(\sqrt{C} + \sqrt{\tfrac{m}{\lambda}}) + 2048\gamma^{-2}\lambda^{-4}h'^2(\sqrt{C} + \sqrt{\tfrac{m}{\lambda}})$$
$$+32\lambda\beta\gamma^{-1}m + 16\beta^2\gamma^{-1}(h(\sqrt{C} + \sqrt{\tfrac{m}{\lambda}}) + ||g||_{G_\sigma^k})^2.$$

We then have,

$$\Gamma(t) + \frac{\gamma}{64}|B^{k-1}e^{\sigma B}\hat{v}_t(t)|_2^2 + \Gamma_1(t) \leq$$
$$\frac{\gamma}{64}\lambda^{-2}m + 64\gamma^{-1}\lambda^{-\frac{5}{2}}\sqrt{C}h'(\sqrt{C} + \sqrt{\tfrac{m}{\lambda}}) + 2048\gamma^{-2}\lambda^{-5}h'^2(\sqrt{C} + \sqrt{\tfrac{m}{\lambda}})$$
$$+\frac{1}{2}m + \frac{1}{256}\gamma(h(\sqrt{C} + \sqrt{\tfrac{m}{\lambda}}) + ||g||_{G_\sigma^k})^2,\ \forall t > 0.$$

As we already bound Γ_1, the desired estimate of $(\hat{v}, \hat{v}_t)$ in $L^\infty(\mathbb{R}^+, \mathcal{F}_\sigma^k(\varepsilon))$ is obtained. □

For λ large enough, we now prove the second part of the theorem 1 (but for smooth intial data), that is, $(w, w_t) = (u^\varepsilon - v, u_t^\varepsilon - v_t)$ goes to zero in $\mathcal{E}^0(\varepsilon)$-norm. The proof is based on energy estimate with $\mathcal{E}^0(\varepsilon)$-norm of (w, w_t) solution of

$$\varepsilon w_{tt} + \alpha w_t - \triangle w + Q_\lambda(F(u^\varepsilon, \hat{v})w) = 0,$$
$$w_{|t=0} = Q_\lambda u_0, \hat{v}_{t|t=0} = Q_\lambda u_1,$$

with

$$F(u^\varepsilon,\hat{v}) = \int_0^1 f'(u^\varepsilon + \theta(P_\lambda u^\varepsilon + \hat{v}))d\theta.$$

We assume again here that (u_0, u_1) belongs to $\mathcal{E}^k(\varepsilon)$, $(k > \frac{5}{2})$, so that we have uniform bound with time of $(u^\varepsilon, u_t^\varepsilon)$ in $\mathcal{E}^k(\varepsilon)$.
We compute the $\mathcal{E}^0(\varepsilon)$ inner product this equation with (w, w_t), we obtain,

$$\begin{gathered}\tfrac{d}{dt}\Gamma + (\gamma - \alpha\varepsilon)|w_t|_2^2 + \alpha|Bw|_2^2 \leq \\ |F(u^\varepsilon,\hat{v})|_\infty |w|_2^2 + \alpha|F(u^\varepsilon,\hat{v})|_\infty |w|_2|w_t|_2 \\ +\beta|\partial_t F(u^\varepsilon,\hat{v})|_\infty |w|_2|A^{-1}w_t|_2 + \beta|F(u^\varepsilon,\hat{v})|_\infty |w|_2|A^{-1}w_t|_2,\end{gathered}$$

with

$$\Gamma = \varepsilon|w_t|_2^2 + |Bw|_2^2 + \alpha|w|_2^2 + \beta|B^{-1}w_t|_2^2 + \beta(Q_\lambda F(u^\varepsilon,\hat{v})w, A^{-1}w_t) + \alpha\varepsilon(w, w_t).$$

Then,

$$\frac{d}{dt}\Gamma + (\frac{\gamma}{2} - \alpha\varepsilon)|w_t|_2^2 + \alpha|Bw|_2^2 \leq C\lambda^{-1}|Bw|_2^2 + \frac{\alpha^2}{\gamma}C^2\lambda^{-1}|Bw|_2^2 + \beta^2C^2\lambda^{-3}|Bw|_2^2,$$

where C depends on bound of F and $\partial_t F$. Choosing, $\alpha = 1$, $\beta = \min(1, \frac{\lambda^2}{2C})$ and λ large enough so that,

$$\lambda \geq 2C + 2C^2(\gamma^{-1} + \lambda^{-2})$$

and $\varepsilon_{\max} \leq \min(\frac{1}{2}, \frac{\gamma}{4})$, we have

$$\Gamma \geq \frac{1}{2}\left(\varepsilon|w_t|_2^2 + |Bw|_2^2 + \alpha|w|_2^2 + \beta|B^{-1}w_t|_2^2\right),$$

$$\Gamma \leq \frac{3}{2}\left(\varepsilon|w_t|_2^2 + |Bw|_2^2 + \alpha|w|_2^2 + \beta|B^{-1}w_t|_2^2\right),$$

and, for $\lambda \geq \beta$,

$$\Gamma(t) \leq \Gamma(0)\exp(-\min(\frac{\gamma}{12}, \frac{1}{3})t).$$

Let us conclude to the proof of theorem 1 with the

Lemma 4. *Let $\varepsilon \leq \varepsilon_{\max}$, let $k > \frac{5}{2}$, let us assume that (u_0, u_1) belongs to $\mathcal{E}^0(\varepsilon)$. There exists $(\tilde{u}(t), \tilde{u}_t(t))$ uniformly bounded with t and ε in $\mathcal{E}^k(\varepsilon)$ such that there exist nonnegative reals m_1 and μ_1, independent of ε such that*

$$||(\tilde{u}(t_0), \tilde{u}_t(t_0)) - S^\varepsilon(t_0)(u_0, u_1)||_{\mathcal{E}^0(\varepsilon)} \leq m_1\exp(-\mu_1 t_0). \tag{13}$$

Assume also $\sigma \leq \sigma_{\max}$, for all $(\tilde{u}_0, \tilde{u}_1)$ in $\mathcal{B} \subset \mathcal{E}^k(\varepsilon)$, there exist (v, v_t) uniformly bounded with respect to ε in $L^\infty(\mathbb{R}^+, \mathcal{F}_\sigma^k(\varepsilon))$ such that there exist nonnegative reals m_2 and μ_2, independent of ε such that

$$||S^\varepsilon(t_1)(\tilde{u}_0, \tilde{u}_1) - (v(t_1), v_t(t_1))||_{\mathcal{E}^0(\varepsilon)} \leq m_2\exp(-\mu_2 t_1),\ \forall t_1 \geq 0, \tag{14}$$

There exist nonnegative reals m_3 and μ_3, independent of ε such that

$$\begin{gathered}||S^\varepsilon(t_1 + t_0)(u_0, u_1) - S^\varepsilon(t_1)(\tilde{u}(t_0), \tilde{u}_t(t_0))||_{\mathcal{E}^0(\varepsilon)} \\ \leq m_3\exp(\mu_3 t_1)||S^\varepsilon(t_0)(u_0, u_1) - (\tilde{u}(t_0), \tilde{u}_t(t_0))||_{\mathcal{E}^0(\varepsilon)},\ \forall t_0, t_1 \geq 0.\end{gathered} \tag{15}$$

Proof of Lemma. The estimate (13) can be found in [2], it is based on the following splitting of $S^\varepsilon(t)$, $u^\varepsilon = d^\varepsilon + r^\varepsilon$,

$$\varepsilon d^\varepsilon_{tt} + \gamma d^\varepsilon_t + A d^\varepsilon + f_1(d^\varepsilon) = 0$$

$$d^\varepsilon_{|t=0} = u^\varepsilon,\ d^\varepsilon_{t|t=0} = 0,$$

$$\varepsilon r^\varepsilon_{tt} + \gamma r^\varepsilon_t + A r^\varepsilon + f_1(d^\varepsilon + r^\varepsilon) - f_1(d^\varepsilon) + f_2(u^\varepsilon) = g$$

$$d^\varepsilon_{|t=0} = 0,\ d^\varepsilon_{t|t=0} = 0,$$

with periodic boundary condition and $f = f_1 + f_2$, $f_1' \geq 0$, $|f_2| + |f_2'| + |f_2''|$ is bounded.

The estimate (14) is what is shown above.

The estimate (15) is a classical estimate of the difference of two solutions. □

Then, choosing $t_1 = \frac{\mu_1}{2\mu_3+\mu_1} t_0$, (13) and (15) lead to

$$||S^\varepsilon(t_1 + t_0)(u_0, u_1) - S^\varepsilon(t_1)(\tilde{u}(t_0), \tilde{u}_t(t_0))||_{\mathcal{E}^0(\varepsilon)}$$
$$\leq m_1 m_3 \exp(-\tfrac{\mu_1}{2}(t_1 + t_0)),\ \forall t_0, t_1 = \tfrac{\mu_1}{2\mu_3+\mu_1} t_0 \geq 0.$$

At last, thanks to (14),

$$||S^\varepsilon(t_1 + t_0)(u_0, u_1) - (v(t_1), v_t(t_1))||_{\mathcal{E}^0(\varepsilon)}$$
$$\leq m_1 m_3 \exp(-\tfrac{\mu_1}{2}(t_1 + t_0)) + m_2 \exp(-\mu_2 t_1),\ \forall t_0, t_1 = \tfrac{\mu_1}{2\mu_3+\mu_1} t_0 \geq 0.$$

This shows the theorem 1.

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