ASYMPTOTIC REGULARITY OF SOLUTIONS OF A NONAUTONOMOUS DAMPED WAVE EQUATION WITH A CRITICAL GROWTH EXPONENT.

S. Zelik

Laboratoire d'Applications des Mathématiques - SP2MI
Boulevard Marie et Pierre Curie - Téléport 2
86962 Chasseneuil Futuroscope Cedex - France

ABSTRACT. The paper is devoted to study of the longtime behavior of solutions of a damped semilinear wave equation in a bounded smooth domain of $\mathbb{R}^3$ with the nonautonomous external forces and with the critical cubic growth rate of the nonlinearity. In contrast to the previous papers, we prove the dissipativity of this equation in higher energy spaces E^α, $0 < \alpha \leq 1$, without the usage of the dissipation integral (which is infinite in our case).

INTRODUCTION.

We study the following damped wave equation in a smooth bounded domain Ω of $\mathbb{R}^3$:

$$(0.1) \qquad \begin{cases} \partial_t^2 u + \gamma \partial_t u - \Delta_x u + f(u) = g(t), & u\big|_{\partial\Omega} = 0, \quad t \geq \tau, \\ u\big|_{t=\tau} = u_\tau, \quad \partial_t u\big|_{t=\tau} = u'_\tau. \end{cases}$$

Here $u = u(t,x)$ is an unknown function, Δ_x is the Laplacian with respect to the variable $x = (x^1, x^2, x^3)$, $\gamma > 0$ is a given dissipation parameter, $\tau \in \mathbb{R}$ and $f = f(u)$ and $g = g(t,x)$ are given nonlinear interaction function and external forces respectively.

We also assume that the nonlinear interaction function $f \in C^2(\mathbb{R})$ has a critical cubic growth rate, i.e.

$$(0.2) \qquad |f''(u)| \leq C(1 + |u|), \quad u \in \mathbb{R}, \quad f(0) = 0$$

and satisfy the standard dissipativity assumption

$$(0.3) \qquad \liminf_{|u| \to \infty} \frac{f(u)}{u} > -\lambda_1,$$

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where λ_1 is the first eigenvalue of the Laplacian in Ω (with the Dirichlet boundary conditions) and the external forces $g(t)$ satisfy

$$(0.4) \qquad\qquad g, \partial_t g \in L^\infty(\mathbb{R}, L^2(\Omega)).$$

It is well-known that, under the above assumptions, equation (0.1) is uniquely solvable in the energy phase space $E := H_0^1(\Omega) \times L^2(\Omega)$, for every $\tau \in \mathbb{R}$ and $\xi_\tau := (u_\tau, u_\tau') \in E$, and, thus, generates a dynamical process in E via

$$(0.5) \qquad\qquad U(t, \tau)\xi_\tau := \xi_u(t), \quad \tau \in \mathbb{R}, \quad t \geq \tau,$$

where $\xi_u(t) := (u(t), \partial_t u(t))$ is a unique solution of (0.1) with the initial data $\xi_\tau \in E$, see [1-2], [4], [9] and the references therein.

It is also well-known that, in the subcritical case

$$(0.6) \qquad\qquad |f''(u)| \leq C(1 + |u|^\kappa), \quad \kappa < 1,$$

problem (0.1) generates a dissipative dynamical process not only in the energy phase space E, but also in more regular phase spaces E^α, $0 \leq \alpha \leq 1$, where

$$(0.7) \qquad\qquad E^\alpha := H^{\alpha+1} \times H^\alpha$$

and $H^s := D((-\Delta_x)^{s/2})$, $s \in \mathbb{R}$, is a scale of Hilbert spaces generated by the Laplacian (equipped by the Dirichlet boundary conditions). Moreover, dynamical process (0.5) associated with equation (0.1) in E possesses a uniformly attracting set bounded in E^1, see [4] for details. In particular, this implies (see [2] and [4]) the existence of a global (uniform) attractor $\mathcal{A}$ for dynamical process (0.5) and its boundedness in more regular space $E^1 = (H^2(\Omega) \cap H_0^1(\Omega)) \times H_0^1(\Omega)$.

In contrast to that, in the case of the critical cubic growth rate, the analogues of the above results were obtained in the autonomous case ($g(t) \equiv g_0 \in L^2(\Omega)$) only (see [2] and [8]) and their proof *essentially* used the finiteness of the so-called dissipation integral

$$(0.8) \qquad\qquad \int_\tau^\infty \|\partial_t u(s)\|_{L^2}^2 \, ds < \infty$$

which is usually infinite in the nonautonomous case (see also [12] where equations of the view (0.1) whose dissipation integral is infinite, but the rate of its divergence is, in a sense, small were considered).

In the present paper, we generalize the method of [2] using instead of the dissipation integral the special approximations of the solution $u(t)$ by the piecewise continuous regular functions (see Proposition 1.4) and, thus, we extend the results mentioned above to the nonautonomous case. To be more precise, the main result of the paper is the following theorem.

Theorem 0.1. *Let assumptions* (0.2)–(0.4) *hold. Then,*

1) For every $\alpha \in [0, 1]$, equation (0.1) generates a dissipative dynamical process $U(t, \tau)$ in the phase space E^α, i.e., for every solution $u(t)$ of this equation satisfying the assumption $\xi_u(\tau) \in E^\alpha$, the following estimate hold:

$$(0.9) \qquad\qquad \|\xi_u(t)\|_{E^\alpha} \leq Q(\|\xi_u(\tau)\|_{E^\alpha})e^{-\mu(t-\tau)} + C,$$

where the positive constants μ and C and the monotonely increasing function Q depend on α, but are independent of t, τ and $\xi_u(\tau)$.

2) The R-ball $\mathbb{B}_R$ of the space E^1 centered at 0 is a uniform exponentially attracting set for dynamical process (0.5) in the phase space E if R is large enough, i.e. there exist a positive constant μ and a monotone increasing function Q such that, for every $\tau \in \mathbb{R}$ and $t \geq 0$ and every bounded set B in E, we have

$$(0.10) \qquad \operatorname{dist}_E\left(U(t+\tau,\tau)B, \mathbb{B}_R\right) \leq Q(\|B\|_E)e^{-\mu t},$$

where $\operatorname{dist}_V(X,Y)$ denotes the non-symmetric Hausdorff distance between the subsets X and Y of the space V.

In particular, Theorem 0.1 implies that the global/uniform attractor $\mathcal{A}$ of problem (0.1) is bounded in E^1. Moreover, estimate (0.10) can be applied in order to construct the *exponential* attractor $\mathcal{M}$ for this problem which will be bounded in E^1 (see e.g., [5]).

We emphasize once more that, in contrast to the previous papers, our method uses neither the Lyapunov function nor the dissipation integral and, thus, can be extended to the class of nongradient damped hyperbolic systems, more general classes of the nonlinearities (e.g., depending explicitly on t) and even to the class of damped hyperbolic equations in unbounded domains (where the dissipation integral is also usually infinite, see e.g. [7] and [11]). We return to these problems somewhere else.

The paper is organized as follows. The proof of Theorem 0.1 is given in Section 1 and some auxiliary results which are necessary for that proof are considered in Appendix.

§1 PROOF OF THEOREM 0.1.

We divide the proof of this theorem on several steps.

Step 1. Dissipativity in E. At this step, extending the arguments of [1] and [6] to the nonautonomous case, we verify that equation (0.1) generates a dissipative process in the energy phase space E.

Proposition 1.1. *Let the assumptions of Theorem 0.1 hold. Then, for every $\tau \in \mathbb{R}$ and $\xi_\tau \in E$, problem (0.1) possesses a unique solution $\xi_u \in C([\tau, +\infty), E)$ and*

$$(1.1) \qquad \|\xi_u(t)\|_E \leq Q(\|\xi_\tau\|_E), \quad t \geq \tau,$$

where the monotonic function Q is independent of t and τ. Moreover, the dynamical process $U(t,\tau) : E \to E$ associated with this equation possesses a uniform bounded absorbing set $\mathbb{B} \subset E$, i.e., for every bounded set $B \subset E$ there exists a time $T = T(B)$ such that

$$(1.2) \qquad U(t+\tau,\tau)B \subset \mathbb{B}, \quad \forall t \geq T, \ \tau \in \mathbb{R}.$$

Proof. The existence and uniqueness of the energy solution $\xi_u(t)$ of equation (0.1) is well-known, so we omit its proof here and only give the formal derivation of

assertions (1.1) and (1.2) which can be justified in a standard way using the Galerkin approximations, see [2] and [9] for the details. To this end, we multiply equation (0.1) by $2(\partial_t u + \beta u)$, where $\beta > 0$ is a sufficiently small positive number which will be specified below, and integrate over Ω. Then, we have

$$(1.3) \quad \frac{d}{dt}\mathcal{E}(\xi_u(t)) + 2(\gamma - \beta)\|\partial_t u(t)\|_{L^2}^2 + 2\beta\|\nabla_x u(t)\|_{L^2}^2 + 2\beta(f(u(t)), u(t)) =$$
$$= 2(g(t), \partial_t u(t) + \beta u(t)),$$

where (u, v) denotes the standard inner product in $L^2(\Omega)$,

$$(1.4) \qquad \mathcal{E}((u, v)) := \|v\|_{L^2}^2 + \|\nabla_x u\|_{L^2}^2 + \beta\gamma\|u\|_{L^2}^2 + 2\beta(u, v) + 2(F(u), 1)$$

and $F(u) := \int_0^u f(v)\, dv$. Moreover, the dissipativity assumption (0.3) implies that, for every $\varepsilon > 0$,

$$(1.5) \qquad 1.\ f(u) \cdot u \geq -(\lambda_1 - \varepsilon)|u|^2 - C_\varepsilon, \quad 2.\ F(u) \geq -\frac{1}{2}(\lambda_1 - \varepsilon)|u|^2 - C_\varepsilon$$

with the appropriate constant C_ε. Using the second estimate of (1.5) and the inequality $\|\nabla_x u\|_{L^2}^2 \geq \lambda_1\|u\|_{L^2}^2$, we deduce that, for sufficiently small $\beta > 0$

$$(1.6) \qquad\qquad\qquad \mathcal{E}(\xi_u(t)) \geq \mu_1\|\xi_u(t)\|_E^2 - C_1,$$

for some positive constants μ_1 and C_1. On the other hand, due to the growth restriction (0.2) and the embedding $H^1 \subset L^6$, we have

$$(1.7) \qquad\qquad\qquad \mathcal{E}(\xi_u(t)) \leq C(\|\xi_u(t)\|_E + 1)^4,$$

for some positive constant C. Moreover, applying the first inequality of (1.5) to equation (1.3) and using the Cauchy-Schwartz inequality, we finally have

$$(1.8) \qquad\qquad \frac{d}{dt}\mathcal{E}(\xi_u(t)) + \delta\|\xi_u(t)\|_E^2 \leq k := C_2(1 + \|g\|_{L^\infty(\mathbb{R}, L^2)}^2),$$

for some positive constants δ and C_2. In order to deduce the desired estimates of $\|\xi_u(t)\|_E$ from the differential inequality (1.8), we need the following lemma.

Lemma 1.1. *Let $\mathcal{E} : E \to \mathbb{R}$ be a continuous semibounded from below functional on a Banach space E. Then, for every $M > 0$, $\varepsilon > 0$ and every function $\xi_u \in C(\mathbb{R}_+, E)$ which satisfies (in the sense of distributions) the differential inequality (1.8) and the additional assumption*
$$\mathcal{E}(\xi_u(0)) \leq M,$$

there exists time $T = T(\varepsilon, M)$ which depends on M and ε, but is independent of a concrete choice of the function $\xi_u(t)$, such that

$$(1.9) \qquad \mathcal{E}(\xi_u(t)) \leq \sup\left\{\mathcal{E}(\xi)\ :\ \xi \in E,\ \delta\|\xi\|_E^2 \leq k + \varepsilon\right\}, \quad \forall t \geq T.$$

The proof of this lemma can be found, e.g. in [3, Lemma 2.7].

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We are now ready to finish the proof of Proposition 1.1. Indeed, applying Lemma 1.1 (with the initial time $t = \tau$ instead of $t = 0$) to the differential inequality (1.8) and using estimates (1.6) and (1.7), we obtain that the set

$$(1.10) \qquad \mathbb{B} := \left\{ \xi \in E \ : \ \mathcal{E}(\xi) \leq \sup \left\{ \mathcal{E}(\eta) \ : \ \eta \in E, \ \delta \|\eta\|_E^2 \leq 2k \right\} \right\}$$

is a bounded uniformly absorbing set for the process $U(t, \tau)$ associated with problem (0.1). Thus, (1.2) is verified. Let us verify (1.1). To this end, we note that it is sufficient to verify this estimate on the finite interval $[\tau, \tau + T]$, where $T = T(\|\xi_\tau\|_E)$ is the same as in (1.2) (since $\xi_u(t) \in \mathbb{B}$ for $t \geq \tau + T$ and $\mathbb{B}$ is bounded). Integrating now inequality (1.8) over $[\tau, \tau + t]$, $t \leq T$, we have

$$(1.11) \qquad \mathcal{E}(\xi_u(t + \tau)) \leq \mathcal{E}(\xi_u(\tau)) + kt.$$

This estimate (together with (1.6) and (1.7)) gives the desired estimate for $\xi_u(t+\tau)$, $t \leq T$ and finishes the proof of Proposition 1.1.

Corollary 1.1. *Let the above assumptions hold and let $u_1(t)$ and $u_2(t)$ be two solutions of problem (0.1). Then, the following estimate holds:*

$$(1.12) \qquad \|\xi_{u_1}(t) - \xi_{u_2}(t)\|_E \leq C e^{K(t-\tau)} \|\xi_{u_1}(\tau) - \xi_{u_2}(\tau)\|_E, \quad t \geq \tau,$$

where the constants C and K depend on $\|\xi_{u_i}(\tau)\|_E$, $i = 1, 2$, but are independent of t, τ and the concrete choice of the solutions u_1 and u_2.

The proof of this estimate is standard and we omit it here, see e.g. [2] and [9].

Assertions (1.1) and (1.2) can be reformulated in a more standard form of a single estimate.

Corollary 1.2. *Let the above assumptions hold. Then, the following estimate hold for every solution $u(t)$ of problem (0.1):*

$$(1.13) \qquad \|\xi_u(t)\|_E \leq \bar{Q}(\|\xi_u(\tau)\|_E) e^{-\alpha(t-\tau)} + C,$$

where the positive constants C and α and the monotonely increasing function $\bar{Q}$ are independent of t, τ and $\xi_u(\tau)$.

Indeed, estimate (1.13) is an obvious corollary of (1.1) and (1.2), thus we omit its proof here and only recall that the constant $\alpha > 0$ in (1.13) can be chosen arbitrarily, C can be specified as the radius of the absorbing ball for the process $U(t, \tau)$ and the function $\bar{Q}$ can be then computed in terms of α, the function Q defined in (1.1) and the function T defined in (1.2).

Thus, estimate (0.9) is verified for $\alpha = 0$ and the first step of the proof of Theorem 0.1 is finished.

Remark 1.1. Dissipativity assumption (0.3) can be replaced by slightly more strong (but, in a sense, more natural) one:

$$(1.14) \qquad \liminf_{|u| \to \infty} f'(u) > -\lambda_1.$$

Indeed, on the one hand, this assumption obviously implies (0.3), but, on the other hand, it is not difficult to deduce from (1.14) that: for every $\varepsilon > 0$, there exists a constant C_ε such that

$$F(u) \leq f(u) \cdot u + \frac{1}{2}(\lambda_1 - \varepsilon)|u|^2 + C_\varepsilon, \quad \forall u \in \mathbb{R}.$$

Estimating the term $(f(u), u)$ in (1.3) by this inequality, we deduce more simple analogue of the differential inequality (1.8):

$$(1.15) \qquad \frac{d}{dt}\mathcal{E}(\xi_u(t)) + \delta\mathcal{E}(\xi_u(t)) \leq k$$

and finish the proof of Proposition 1.1 in a standard way (see [2] and [9]) applying the Gronwall's inequality to this relation (without using Lemma 1.1).

It is also worth to emphasize that the function f which satisfies (1.14) and (0.2) automatically satisfies the assumptions of Babin and Vishik, see [2, Sections I.8 and II.6].

Step 2.   Dissipativity in E^α with $0 < \alpha < 1/2$. At this step, we prove estimate (0.9) for $0 < \alpha < 1/2$ and construct an exponentially attracting set which is bounded in E^α. To this end, following [2], we split the solution $u(t)$ of equation (0.1) as follows: $u(t) = v(t) + w(t)$, where $v(t)$ solves the following autonomous problem:

$$(1.16) \qquad \partial_t^2 v + \gamma\partial_t v - \Delta_x v + f(v) + Lv = 0, \quad \xi_v\big|_{t=\tau} = \xi_u\big|_{t=\tau}, \quad t \geq \tau,$$

where L is a sufficiently large positive number and the remainder $w(t)$ satisfies:

$$(1.17) \qquad \partial_t^2 w + \gamma\partial_t w - \Delta_x w + [f(v + w) - f(v)] = g(t) + Lv(t), \quad \xi_w\big|_{t=\tau} = 0.$$

We first study equation (1.16).

Proposition 1.2. *Let the above assumptions hold. Then, there exists a positive constant L such that the solution $\xi_v(t)$ of (1.16) satisfies*

$$(1.18) \qquad \|\xi_v(t)\|_E \leq Q(\|\xi_u(\tau)\|_E)e^{-\mu(t-\tau)},$$

where the positive constant μ and monotonely increasing function Q are independent of t, τ and $\xi_u(\tau)$.

Proof. We first note that it is sufficient to prove estimate (1.18) for $\tau = 0$ only (since equation (1.16) is autonomous). We also note that the dissipativity assumption (0.3) and the fact that $f(0) = 0$ imply that

$$(1.19) \qquad 1. \ f(u) \cdot u \geq -K|u|^2, \quad 2. \ F(u) \geq -\frac{K}{2}|u|^2,$$

for some positive K. Then, multiplying equation (1.16) by $\partial_t v + \beta v$, integrating over Ω and arguing as in derivation of (1.8) but using (1.19) instead of (1.5), we obtain that, for $L > K$,

$$(1.20) \qquad \frac{d}{dt}\mathcal{E}_L(\xi_v(t)) + \delta\|\xi_v(t)\|_E \leq 0,$$

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where δ is a positive constant and

(1.21) $\mathcal{E}_L((u,v)) := \|v\|_{L^2}^2 + \|\nabla_x u\|_{L^2}^2 + \beta\gamma\|u\|_{L^2}^2 + 2\beta(u,v) + 2(F(u),1) + L\|u\|_{L^2}^2.$

Moreover, due to estimate (1.19)(2) and the fact that $L > K$, we have the following improved analogue of (1.6):

(1.22) $$\mathcal{E}_L(\xi_v(t)) \geq \mu_1\|\xi_v(t)\|_E^2,$$

for some positive μ_1. Thus, applying Lemma 1.1 to the differential inequality (1.20) and using (1.7) and (1.22), we verify that, for every bounded subset $B \subset E$,

(1.23) $$\lim_{t\to\infty}\sup\{\|\xi_v(t)\|_E \ : \ \xi_v(0) \in B\} = 0$$

and, consequently, every trajectory of equation (1.16) converges (uniformly with respect to the initial data belonging to bounded subsets) to the equilibrium $u \equiv 0$ of this equation. There remains to note that this equilibrium is locally exponentially stable (since $L > f'(0)$, see e.g. [2]) and, therefore, the rate of convergence in (1.23) is, in a fact, exponential (i.e., (1.18) holds) and Proposition 1.2 is proven.

We now study equation (1.17).

Proposition 1.3. *Let the above assumptions hold. Then, for every bounded subset $B \subset E$ and every $0 < \alpha < 1/2$, there exist positive constants C and K (depending only on B and α) such that, for every $\tau \in \mathbb{R}$ and every $\xi_u(\tau) \in B$, the following estimate is valid:*

(1.24) $$\|\xi_w(t)\|_{E^\alpha} \leq Ce^{K(t-\tau)}, \quad t \geq \tau.$$

Proof. Differentiating equation (1.17) and setting $\theta(t) := \partial_t w(t)$, we have

(1.25) $\partial_t^2\theta + \gamma\partial_t\theta - \Delta_x\theta = -[f'(v+w) - f'(v)]\partial_t u - f'(v)\theta + g'(t) + L\partial_t v(t).$

Moreover, expressing the second derivative of $w(t)$ from equation (1.17) and taking into account that $\xi_w(0) = 0$ and growth restriction (0.2), we have

(1.26) $\theta(\tau) = 0, \ \partial_t\theta(\tau) = -f(u(\tau)) + g(\tau) + Lu(\tau), \ \|\partial_t\theta(\tau)\|_{L^2} \leq Q_1(\|\xi_u(\tau)\|_E),$

for some monotonely increasing function Q_1.

Let us now fix $0 < \alpha < 1/2$, multiply equation (1.25) by $(-\Delta_x)^{\alpha-1}(\partial_t\theta + \beta\theta)$ (where $\beta > 0$ is small enough) and integrate over Ω. Then, after the standard transformations, we have

(1.27) $\dfrac{1}{2}\dfrac{d}{dt}\tilde{\mathcal{E}}(\xi_\theta(t)) + \delta\tilde{\mathcal{E}}(\xi_\theta(t)) \leq$

$$\leq -([f'(v+w) - f'(v)]\partial_t u, (-\Delta_x)^{\alpha-1}(\partial_t\theta + \beta\theta)) -$$
$$- ([f'(v) - f'(0)]\theta, (-\Delta_x)^{\alpha-1}(\partial_t\theta + \beta\theta)) +$$

$+ (g'(t) + L\partial_t v(t) - f'(0)(\partial_t u(t) - \partial_t v(t)), (-\Delta_x)^{\alpha-1}(\partial_t\theta + \beta\theta)) := I_1 + I_2 + I_3,$

where

(1.28) $$\tilde{\mathcal{E}}(\xi_\theta(t)) := \|\xi_\theta(t)\|_{E^{\alpha-1}}^2 + \beta\gamma\|\theta(t)\|_{H^{\alpha-1}}^2 + 2\beta(\theta(t), \partial_t\theta(t))_{H^{\alpha-1}}$$

and $\beta > 0$ is small enough. Thus, we need to estimate the integrals I_1, I_2 and I_3. We first note that, due to (0.4), (1.1) and (1.18), the integral I_3 can be estimated as follows

(1.29) $$I_3 \leq C_\varepsilon + \varepsilon\|\xi_\theta(t)\|_{E^{\alpha-1}}^2 \leq C_\varepsilon + 2\varepsilon\tilde{\mathcal{E}}(\xi_\theta(t)),$$

where $\varepsilon > 0$ is arbitrary and C_ε depends on ε and on the bounded subset $B \subset E$.

In order to estimate the integrals I_1 and I_2, we need the following lemma.

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Lemma 1.2. *Let $0 \leq \alpha < 1/2$. Then*

(1.30)
$$\begin{cases} 1. & \|u_1 \cdot (-\Delta_x)^{\alpha-1} u_2\|_{L^3} \leq C\|u_1\|_{H^{1+\alpha}} \|u_2\|_{H^{\alpha-1}}, \\ 2. & \|u_3 \cdot (-\Delta_x)^{\alpha-1} u_2\|_{L^{3/2}} \leq C\|u_3\|_{H^{\alpha}} \|u_2\|_{H^{\alpha-1}}, \end{cases}$$

for all $u_1 \in H^{\alpha+1}$, $u_2 \in H^{\alpha-1}$ and $u_3 \in H^{\alpha}$ and for some constant C which depends on α, but is independent of u_1, u_2 and u_3.

The proof of Lemma 1.2 is given in Appendix.

Let us now estimate the integral I_1. To this end, we first note that the growth restriction (0.2) implies the following estimate

(1.31)
$$|f'(v+w) - f(v)| \leq C|w| \cdot (|v| + |v+w|), \quad \forall v, w \in \mathbb{R},$$

where the constant C is independent of v and w. Moreover, expressing the term $\Delta_x w$ from equation (1.17) and taking the $H^{\alpha-1}$-norm from the both sides of the equation obtained, we have

(1.32) $\quad \|w(t)\|_{H^{\alpha+1}} \leq \|\partial_t^2 w(t)\|_{H^{\alpha-1}} + \gamma\|\partial_t w(t)\|_{L^2} + \|f(u(t))\|_{L^2} +$
$$+ \|f(v(t))\|_{L^2} + \|g(t)\|_{L^2} + L\|v(t)\|_{L^2} \leq \|\partial_t \theta(t)\|_{H^{\alpha-1}} + C_1,$$

where the constant C_1 depends on the bounded subset B, but is independent of t and τ (here we have implicitly used estimates (1.1) and (1.18) and the embedding $H^1 \subset L^6$). Applying Hölder inequality with exponents 6, 3 and 2 to the integral I_2 and using inequalities (1.31) and (1.32) and the first inequality of (1.30), we obtain

(1.33) $\quad I_1 \leq C(\|u(t)\|_{L^6} + \|v(t)\|_{L^6}) \cdot \|\partial_t u(t)\|_{L^2} \|w \cdot (-\Delta_x)^{\alpha-1}(\partial_t \theta + \beta\theta)\|_{L^3}$
$$\leq C_1 \|\partial_t u(t)\|_{L^2} \|w(t)\|_{H^{\alpha+1}} \|\partial_t \theta(t) + \beta\theta(t)\|_{H^{\alpha-1}} \leq$$
$$\leq C_2 \|\partial_t u(t)\|_{L^2} \|\xi_\theta(t)\|_{E^{\alpha-1}}^2 + C_3 \leq C_\varepsilon \|\partial_t u(t)\|_{L^2}^2 \tilde{\mathcal{E}}(\xi_\theta(t)) + \varepsilon\tilde{\mathcal{E}}(\xi_\theta(t)) + C_3,$$

where ε is an arbitrary positive constant and the constants C_ε, C and C_i depend on the bounded subset $B \subset E$.

Finally, applying Hölder inequality with the exponents 3 and 3/2 to the integral I_2 and using the second estimate of (1.30), we have

(1.34) $\quad I_2 \leq \|f'(v(t)) - f'(0)\|_{L^3} \|\theta \cdot (-\Delta_x)^{\alpha-1}(\partial_t \theta(t) + \beta\theta(t))\|_{L^{3/2}} \leq$
$$\leq C_1 \|f'(v(t)) - f'(0)\|_{L^3} \|\theta(t)\|_{H^\alpha} \|\partial_t \theta(t) + \beta\theta(t)\|_{H^{\alpha-1}} \leq$$
$$\leq C_2 \|f(v(t)) - f'(0)\|_{L^3} \|\xi_\theta(t)\|_{E^{\alpha-1}}^2 \leq 2C_2 \|f'(v(t)) - f'(0)\|_{L^3} \tilde{\mathcal{E}}(\xi_\theta(t)),$$

where the constants C_i depend on the bounded subset $B \subset E$. Inserting estimates, (1.29), (1.33) and (1.34) into the right-hand side of (1.27) and fixing $\varepsilon > 0$ small enough, we deduce that

(1.35) $\quad \dfrac{d}{dt} \tilde{\mathcal{E}}(\xi_\theta(t)) + [\delta - C(\|\partial_t u(t)\|_{L^2}^2 + \|f'(v(t)) - f'(0)\|_{L^3})]\tilde{\mathcal{E}}(\xi_\theta(t)) \leq M,$

for some positive constants C and M depending on the bounded set B of the initial data allowed, but are independent of t and τ. Moreover, due to the growth restriction (0.2), the embedding $H^1 \subset L^6$ and estimate (1.18), we have

(1.36) $\quad \|f'(v(t)) - f'(0)\|_{L^3} \leq C\|v(t)\|_{H^1}(1 + \|v(t)\|_{H^1}) \leq Q(\|\xi_u(\tau)\|_E)e^{-\mu(t-\tau)},$

where the positive constant μ and the monotonely increasing function Q are independent of t, τ and $\xi_u(\tau)$.

Applying finally the Gronwall's inequality to (1.35) and using (1.1) in order to estimate the L^2-norm of $\partial_t u$, we infer

$$(1.37) \qquad \|\xi_\theta(t)\|^2_{E^{\alpha-1}} \le 2\tilde{\mathcal{E}}(\xi_\theta(t)) \le M' + 2\tilde{\mathcal{E}}(\xi_\theta(\tau))e^{2K(t-\tau)},$$

for some positive constants M' and K depending only on B. Estimate (1.37), together with (1.26) and (1.32) imply (1.24) and finish the proof of Proposition 1.3.

Corollary 1.3. *Let the above assumptions hold and let, in addition, $\xi_u(\tau) \in E^\alpha$, for some $0 \le \alpha < 1/2$. Then, the following estimate holds:*

$$(1.38) \qquad \|\xi_u(t)\|_{E^\alpha} \le C e^{K(t-\tau)},$$

where the positive constants C and K depend on $\|\xi_u(\tau)\|_{E^\alpha}$, but are independent of t and τ.

The proof of estimate (1.38) is analogous to the proof of Proposition 1.3, but essentially more simple, since now the initial data belong to E^α from the very beginning and we need not now to split the solution $u(t)$ by (1.16) and (1.17) and can directly differentiate equation (0.1) by t and set $\theta(t) = \partial_t u(t)$. Then, we obtain equation (1.25) (with $L = 0$ and $v(t) \equiv 0$), but with different initial data:

$$\theta(\tau) = \partial_t u(\tau), \quad \partial_t \theta(\tau) = \Delta_x u(\tau) - \gamma \partial_t u(\tau) - f(u(\tau)) + g(\tau).$$

Thus, it is not difficult to show, analogously to (1.26) and (1.32), that

$$(1.39) \qquad \|\xi_\theta(\tau)\|_{E^{\alpha-1}} \le Q\left(\|\xi_u(\tau)\|_{E^\alpha}\right),$$

for some monotone increasing function Q. Estimate (1.38) is now an immediate corollary of (1.37), (1.39) and (1.32) and Corollary 1.3 is proven.

Remark 1.2. It is worth to note that the estimate for the E^α-norm of $\xi_w(t)$ obtained in Proposition 1.3 diverges exponentially as $t \to +\infty$. In contrast to that, *in the autonomous case*, differential inequality (1.35) allows to obtain nondivergent estimate for the $\xi_w(t)$ which, in a fact, finishes the proof of Theorem 0.1 for $0 \le \alpha < 1/2$. Indeed, it follows from (1.36) that

$$(1.40) \qquad \int_\tau^\infty \|f(v(t)) - f'(0)\|_{L^3}\, dt \le C < \infty.$$

Moreover, in the autonomous case we also have the dissipation integral (0.8), consequently, the Gronwall's inequality applied to (1.35) gives the non-divergent estimate

$$(1.41) \qquad \tilde{\mathcal{E}}(\xi_\theta(t)) \le C\tilde{\mathcal{E}}(\xi_\theta(\tau))e^{-\mu t} + M',$$

for some positive constants C, M' and μ. Unfortunately, the dissipation integral (0.8) usually equals infinity in the nonautonomous case, thus, the scheme of [2] (described above) now gives the *exponentially divergent* estimates only which is obviously not enough for proving Theorem 0.1.

The following proposition, which gives a splitting of the function $\partial_t u(t)$ in a sum of two functions one of which is regular and the other is, in a sense, small, is a crucial point of our method.

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Proposition 1.4. *Let the above assumptions hold. Then, for every $\mu > 0$, $0 \leq \alpha < 1/2$ and every bounded subset $B \subset E$, there exist positive constants C_μ and K_μ such that, for every solution $u(t)$ of problem (0.1) satisfying $\xi_u(\tau) \in B$, there exists a splitting*

(1.42)
$$\partial_t u(t) = v_1(t) + w_1(t), \quad t \geq \tau$$

such that

(1.43)
$$\|w_1(t)\|_{H^{\alpha+1}} \leq K_\mu$$

and, for every $t \geq s \geq \tau$,

(1.44)
$$\int_s^t \|v_1(\kappa)\|_{L^2}^2 \, d\kappa \leq \mu(t-s) + C_\mu.$$

Proof. In order to construct the functions v_1 and w_1, we fix a large $T > 0$ and, at every interval $[\tau + (n-1)T, \tau + nT]$, we set

$$v_1(t) := \partial_t v(t), \quad w_1(t) := \partial_t w(t),$$

where the functions $v(t)$ and $w(t)$ solve equations (1.16) and (1.17) respectively at the interval $[\tau + (n-1)T, \tau + nT]$, $n \in \mathbb{N}$, with the following initial data:

(1.45)
$$\xi_v(\tau + (n-1)T) := \xi_u(\tau + (n-1)T), \quad \xi_w(\tau + (n-1)T) = 0.$$

Then, according to estimate (1.18), we have

(1.46)
$$\int_{\tau+(n-1)T}^{\tau+nT} \|v_1(\kappa)\|_{L^2}^2 \, d\kappa \leq C,$$

where the constant $C = C(B)$ is independent of τ, n and T. Thus, for every $\mu > 0$, we can find a sufficiently large $T = T(\mu, B)$ such that (1.44) is satisfied. After fixing the length T, estimate (1.24) implies (1.43) for some $K_\mu = K(B, T)$ and finishes the proof of Proposition 1.4.

Our next task is to obtain the non-divergent analogue of estimate (1.24) using splitting (1.42) instead of the dissipation integral.

Proposition 1.5. *Let the above assumptions hold. Then, the solution $\xi_w(t)$ of equation (1.17) possesses the following estimate:*

(1.47)
$$\|\xi_w(t)\|_{E^\alpha} \leq Q_\alpha(\|\xi_u(\tau)\|_E), \quad t \geq \tau,$$

where the monotonely increasing function Q_α depends on $0 \leq \alpha < 1/2$, but is independent of t and τ.

Proof. Analyzing the proof of Proposition 1.3, we see that the exponential divergence in (1.24) appears due to the term $\|\partial_t u(t)\|_{L^2}^2$ in differential inequality (1.35) which, in turns, appears under the estimating of the integral I_1 by (1.33). Thus, our

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task is to improve estimate (1.33) using splitting (1.42). To this end, we transform this integral as follows:

$$(1.48) \quad I_1 = -([f'(v+w) - f'(v)]v_1, (-\Delta_x)^{\alpha-1}(\partial_t\theta + \beta\theta)) -$$
$$- ([f'(v+w) - f'(v)]w_1, (-\Delta_x)^{\alpha-1}(\partial_t\theta + \beta\theta)) := I_1^1 + I_1^2,$$

where the functions v_1 and w_2 are the same as in Proposition 1.4 (with a sufficiently small parameter μ which will be fixed below). Then, arguing exactly as in (1.33), we have

$$(1.49) \qquad I_1^1 \le C_\varepsilon \|v_1(t)\|_{L^2}^2 \tilde{\mathcal{E}}(\xi_\theta(t)) + \varepsilon\tilde{\mathcal{E}}(\xi_\theta(t)) + C,$$

where the constant $\varepsilon > 0$ can be chosen arbitrarily. Applying now Hölder inequality with the exponents 3 and 3/2 to the integral I_1^2 and using the second estimate of (1.30) and estimate (1.43), we infer

$$(1.50) \quad I_1^2 \le (\|f'(u(t))\|_{L^3} + \|f'(v(t))\|_{L^3})\|w_1(t) \cdot (-\Delta_x)^{\alpha-1}(\partial_t\theta(t) + \beta\theta(t))\|_{L^{3/2}}$$
$$\le C_1\|w_1(t)\|_{H^\alpha}\|\partial_t\theta(t) + \beta\theta(t)\|_{H^{\alpha-1}} \le C_2\|\xi_\theta(t)\|_{E^\alpha} \le \varepsilon\tilde{\mathcal{E}}(\xi_\theta(t)) + C_\varepsilon,$$

where the constant $\varepsilon > 0$ can be chosen arbitrarily and the constant C_ε depends on $\|\xi_u(\tau)\|_E$, but is independent of t and τ.

Using now estimates (1.49) and (1.50) instead of (1.33), we can improve differential inequality (1.35) as follows:

$$(1.51) \qquad \frac{d}{dt}\tilde{\mathcal{E}}(\xi_\theta(t)) + h(t)\tilde{\mathcal{E}}(\xi_\theta(t)) \le M,$$

where

$$h(t) := \delta - C(\|v_1(t)\|_{L^2}^2 + \|f'(v(t)) - f'(0)\|_{L^3}),$$

and the positive constants δ, C and M are independent of t and τ. Moreover, fixing $\mu := \delta/(2C)$ in (1.44) and using (1.40), we have

$$(1.52) \qquad \int_s^t h(\kappa)\,d\kappa \ge \frac{1}{2}\delta(t-s) - \bar{C}, \quad t \ge s \ge \tau,$$

where the constant $\bar{C}$ depends on $\|\xi_u(\tau)\|_E$, but is independent of t, s and τ. Applying the Gronwall's inequality to (1.51) and using (1.52), we infer the following improved version of (1.37):

$$(1.53) \qquad \|\xi_\theta(t)\|_{E^{\alpha-1}}^2 \le 2\tilde{\mathcal{E}}(\xi_\theta(t)) \le M' + 2\tilde{\mathcal{E}}(\xi_\theta(\tau))e^{\bar{C}-\delta(t-\tau)/2}.$$

Estimate (1.53), together with (1.26) and (1.32) imply (1.47) and finish the proof of Proposition 1.5.

Corollary 1.4. *Let the above assumptions hold. Then, for every $0 \le \alpha < 1/2$, there exist positive constants R_α and μ and a monotonely increasing function Q such that, for every bounded subset B of E, we have*

$$(1.54) \qquad \mathrm{dist}_E\left(U(t,\tau)B, \{\xi \in E^\alpha : \|\xi\|_{E^\alpha} \le R_\alpha\}\right) \le Q(\|B\|_E)e^{-\mu(t-\tau)},$$

for all $\tau \in \mathbb{R}$ and $t \ge \tau$.

Indeed, due to Proposition 1.1, it is sufficient to verify (1.54) for the absorbing set $\mathbb{B}$ only. But, in this case, estimate (1.54) is an immediate corollary of (1.18) and (1.47) (we can set $R_\alpha := Q_\alpha(\|\mathbb{B}\|_E)$, where Q_α is the same as in (1.47)).

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Corollary 1.5. *Let the above assumptions hold and let, in addition, $\xi_u(\tau) \in E^\alpha$, for some $0 \le \alpha < 1/2$. Then, the following estimate is valid:*

$$(1.55) \qquad \|\xi_u(t)\|_{E^\alpha} \le Q_\alpha(\|\xi_u(\tau)\|_{E^\alpha})e^{-\mu(t-\tau)} + C_\alpha,$$

where the positive constants μ and C_α and the monotonely increasing function Q_α are independent of t, τ and $\xi_u(\tau)$.

Indeed, due to Proposition 1.1 and Corollary 1.3, it is sufficient to verify (1.55), for the initial data belonging to the absorbing set $\mathbb{B}$ only. In this case, estimate (1.55) can be verified analogously to the proof of Corollary 1.3, but using more strong estimate (1.53) instead of (1.37).

Thus, the second step of the proof of Theorem 0.1 is also finished.

Step 3. The case $1/2 \le \alpha \le 1$. At this step, we verify the dissipativity of the dynamical process $U(t,\tau)$ in the spaces E^α, $1/2 \le \alpha \le 1$ and, thus, finish the proof of Theorem 0.1. To this end, it is convenient to use more simple (than (1.16) and (1.17)) splitting of the solution $u(t)$ where the first equation is linear, namely, we set $u(t) := v(t) + w(t)$, where the function $v(t)$ solves

$$(1.56) \qquad \partial_t^2 v + \gamma \partial_t v - \Delta_x v = 0, \quad \xi_v\big|_{t=\tau} = \xi_u\big|_{t=\tau}$$

and the remainder $w(t)$ satisfies

$$(1.57) \qquad \partial_t^2 w + \gamma \partial_t w - \Delta_x w = h_u(t) := g(t) - f(u(t)), \quad \xi_w\big|_{t=\tau} = 0.$$

Then, applying the E^α-regularity estimate for the damped linear equation (1.56) (see e.g., [9]), we infer that, for every $0 \le \alpha \le 1$,

$$(1.58) \qquad \|\xi_v(t)\|_{E^\alpha} \le C\|\xi_u(\tau)\|_{E^\alpha} e^{-\mu(t-\tau)}, \quad t \ge \tau,$$

where the positive constants C and μ are independent of t, τ and $\xi_u(\tau) \in E^\alpha$. Thus, it only remains to study equation (1.57).

Proposition 1.6. *Let the above assumptions hold and let, in addition, $\xi_u(\tau) \in E^{1/3}$. Then, the solution $w(t)$ of equation (1.57) satisfies the following estimate:*

$$(1.59) \qquad \|\xi_w(t)\|_{E^1} \le Q(\|\xi_u(\tau)\|_{E^{1/3}})e^{-\mu(t-\tau)} + C^*, \quad t \ge \tau,$$

where the positive constants μ and C^* and the monotonely increasing function Q are independent of t, τ and $\xi_u(\tau) \in E^{1/3}$.

Proof. According to the E^1-regularity theorem for damped linear wave equations (see e.g., [9]), it is sufficient to verify the following estimate:

$$(1.60) \qquad \|h_u(t)\|_{L^2} + \|\partial_t h_u(t)\|_{L^2} \le Q(\|\xi_u(\tau)\|_{E^\alpha})e^{-\mu(t-\tau)} + C^*.$$

Moreover, due to assumptions (0.2) and (0.4) and estimate (1.13), it is only sufficient to verify that

$$(1.61) \qquad \|f'(u(t))\partial_t u(t)\|_{L^2} \le Q(\|\xi_u(\tau)\|_{E^\alpha})e^{-\mu(t-\tau)} + C^*.$$

In order to verify this estimate, we need to use the fact that $\xi_u(\tau) \in E^{1/3}$ and estimate (1.55) with $\alpha = 1/3$. Indeed, due to this estimate and embeddings $H^{1/3} \subset L^{18/7}$ and $H^{4/3} \subset L^{18}$, we have the desired estimates for the $\|u(t)\|_{L^{18}}$ and $\|\partial_t u(t)\|_{L^{18/7}}$. Moreover, since, due to the growth restriction (0.2), the function $f'(u)$ has a quadratic growth rate, then we also have the desired estimate for $\|f'(u(t))\|_{L^9}$. Since $\frac{1}{9} + \frac{7}{18} = \frac{1}{2}$, then the Hölder inequality gives (1.61) and finishes the proof of Proposition 1.6.

We are now ready to finish the proof of the first part of Theorem 0.1.

Corollary 1.6. *Let the above assumptions hold. Then, for every $1/3 \leq \alpha \leq 1$, the solution $u(t)$ of problem (0.1) satisfies the following estimate:*

$$(1.62) \qquad \|\xi_u(t)\|_{E^\alpha} \leq Q\left(\|\xi_u(\tau)\|_{E^\alpha}\right)e^{-\mu(t-\tau)} + C^*, \quad t \geq \tau,$$

where the positive constants μ and C^* and the monotonely increasing function Q are independent of t, τ and $\xi_u(\tau) \in E^\alpha$.

Indeed, (1.62) is an immediate corollary of (1.58) and (1.55).

Combining Corollaries 1.5 and 1.6, we obtain estimate (0.9) for any $0 \leq \alpha \leq 1$ and finish the proof of the first part of Theorem 0.1.

Corollary 1.7. *Let the above assumptions hold. Then, there exist positive constants R and μ and a monotonely increasing function Q such that, for every bounded subset $B \subset E^{1/3}$, we have*

$$(1.63) \qquad \mathrm{dist}_{E^{1/3}}\left(U(t,\tau)B, \left\{\xi \in E^1 : \|\xi\|_{E^1} \leq R\right\}\right) \leq Q\left(\|B\|_{E^{1/3}}\right)e^{-\mu(t-\tau)},$$

for all $\tau \in \mathbb{R}$ and $t \geq \tau$.

Indeed, estimate (1.63) is also an immediate corollary of (1.58) and (1.55).

In order to verify estimate (0.10), we use the following general fact on the transitivity of an exponential attraction established in [5].

Lemma 1.3. *Let $(\mathcal{M}, d)$ be an abstract metric space and let $U(t, \tau)$ be a Lipschitz continuous dynamical process in $\mathcal{M}$, i.e*

$$(1.64) \qquad d(U(t+\tau,\tau)m_1, U(t+\tau,\tau)m_2) \leq Ce^{Kt}d(m_1, m_2),$$

for appropriate constants C and K which are independent of m_i, τ and t. We further assume that there exist three subsets $M_1, M_2, M_3 \subset \mathcal{M}$ such that

$$(1.65) \qquad \begin{cases} \mathrm{dist}_{\mathcal{M}}\left(U(t+\tau,\tau)M_1, M_2\right) \leq C_1 e^{-\alpha_1 t}, \\ \mathrm{dist}_{\mathcal{M}}\left(U(t+\tau,\tau)M_2, M_3\right) \leq C_2 e^{-\alpha_2 t}. \end{cases}$$

Then

$$(1.66) \qquad \mathrm{dist}_{\mathcal{M}}\left(U(t+\tau,\tau)M_1, M_3\right) \leq C'e^{-\alpha' t},$$

where $C' = CC_1 + C_2$, $\alpha' = \frac{\alpha_1 \alpha_2}{K + \alpha_1 + \alpha_2}$.

The proof of Lemma 1.3 is given in [5] (in the autonomous case). Nevertheless, for the convenience of the reader, we recall it in Appendix.

We are now ready to verify (0.10). Indeed, according to Proposition 1.1, it is sufficient to verify this estimate for the absorbing set $\mathbb{B}$ only (we set $\mathcal{M}_1 := \mathbb{B}$). Then, due to Corollary 1.4, the set $\mathcal{M}_1$ is attracted exponentially to the ball $\mathcal{M}_2$ of radius $R_{1/3}$ of the space $E^{1/3}$. Moreover, due to Corollary 1.7, the ball $\mathcal{M}_2$ is attracted exponentially to the ball $\mathcal{M}_3$ of radius R of the space E^1 (even in a more strong topology of $E^{1/3}$). The uniform Lipschitz continuity of the process $U(t, \tau)$ (on bounded subsets of E) is given by Corollary 1.1. Thus, estimate (0.10) now follows from the transitivity of exponential attraction (Lemma 1.2) and, consequently, Theorem 0.1 is proven.

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In this Appendix we give the proofs of Lemmata 1.2 and 1.3.

Proof of Lemma 1.2. Let us verify the first estimate of (1.30). Indeed, according to the embedding theorem, we have

$$(A.1) \qquad \|u_1\|_{L^{p_1}} \leq C\|u_1\|_{H^{\alpha+1}}, \quad \text{where } 0 \leq \alpha < 1/2 \text{ and } \quad \frac{1}{p_1} = \frac{1}{2} - \frac{1+\alpha}{3}.$$

On the other hand, according to the regularity theorem for the fractional powers of the Laplacian (see e.g. [10]), we have

$$(A.2) \qquad \|(-\Delta_x)^{\alpha-1}u_2\|_{H^{1-\alpha}} \leq C_1\|u_2\|_{H^{\alpha-1}}.$$

Applying again the embedding theorem, we infer

$$(A.3) \qquad \|(-\Delta_x)^{\alpha-1}u_2\|_{L^{p_2}} \leq C_2\|(-\Delta_x)^{\alpha-1}u_2\|_{H^{1-\alpha}} \leq C_3\|u_2\|_{H^{\alpha-1}},$$

where $\frac{1}{p_2} = \frac{1}{2} - \frac{1-\alpha}{3}$. Since

$$\frac{1}{p_1} + \frac{1}{p_2} = \frac{1}{2} - \frac{1+\alpha}{3} + \frac{1}{2} - \frac{1-\alpha}{3} = \frac{1}{3},$$

then, due to Hölder inequality, inequalities (A.1) and (A.3) imply the first estimate of (1.30).

Let us now verify the second inequality of (1.30). Indeed, according to the embedding theorem

$$(A.4) \qquad \|u_3\|_{L^{p_3}} \leq C_4\|u_3\|_{H^\alpha}, \quad \text{where} \quad \frac{1}{p_3} = \frac{1}{2} - \frac{\alpha}{3}.$$

Since $\frac{1}{p_3} + \frac{1}{p_2} = \frac{1}{2} - \frac{\alpha}{3} + \frac{1}{2} - \frac{1-\alpha}{3} = \frac{2}{3}$, then (due to Hölder inequality) estimates (A.3) and (A.4) imply the second estimate of (1.30) and finish the proof of Lemma 1.2.

Proof of Lemma 1.3. Let $\tau \in \mathbb{R}$ be fixed, m_1 belong to M_1 and let us set $t = t_1 + t_2$, where $t_i \geq 0$, $i = 1, 2$, will be fixed below. Then, owing to the first estimate of (1.65), there exists $m_2 \in M_2$ such that

$$(A.5) \qquad d(U(t_1 + \tau, \tau)m_1, m_2) \leq C_1 e^{-\alpha_1 t_1}.$$

Then, estimate (1.64) (and the identity $U(t,\tau) = U(t,s) \circ U(s,\tau)$ for $t \geq s \geq \tau$) implies that

$$(A.6) \qquad d(U(t+\tau, \tau)m_1, U(t+\tau, t_1+\tau)m_2) \leq CC_1 e^{Kt_2 - \alpha_1 t_1}.$$

On the other hand, using the second estimate of (1.65), we deduce that there exists $m_3 \in M_3$ such that

$$(A.7) \qquad d(U(t+\tau, t_1+\tau)m_2, m_3) \leq C_2 e^{-\alpha_2 t_2}.$$

Combining (A.5)–(A.7) and noting that $m_1 \in M_1$ and $t_1 \in [0,t]$ is arbitrary, we obtain

$$(A.8) \qquad \text{dist}_{\mathcal{M}}(U(t+\tau, \tau)M_1, M_3) \leq \inf_{t_1+t_2=t}\left(CC_1 e^{Kt_2-\alpha_1 t_1} + C_2 e^{-\alpha_2 t_2}\right).$$

Fixing the values t_i in an optimal way (i.e. such that $Kt_1 - \alpha_1 t_2 = \alpha_2 t_2$), we obtain (1.66). Lemma 1.3 is proven.

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