THE ATTRACTORS OF REACTION-DIFFUSION SYSTEMS IN UNBOUNDED DOMAINS AND THEIR SPATIAL COMPLEXITY.

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Abstract. The nonlinear reaction diffusion system in an unbounded domain is studied. It is proved that under some natural assumptions on the nonlinear term and on the diffusion matrix this system possesses a global attractor $\mathcal{A}$ in the corresponding phase space. Since the dimension of the attractor is occurred to be infinite we study the Kolmogorov's ε-entropy of it. The upper and lower bounds of this entropy are obtained.

Moreover, we give a more detailed study of the attractor for the spatially homogeneous RDE in $\mathbb{R}^n$. In this case a group of spatial shifts acts on the attractor. In order to study the spatial complexity of the attractor we interpret this group as a dynamical system (with multidimensional 'time' if $n > 1$) acting on a phase space $\mathcal{A}$. It is proved that the dynamical system thus obtained is chaotic and has the infinite topological entropy.

In order to clarify the nature of this chaotisity we suggest a new model dynamical system which generalizes the symbolic dynamics to the case of the infinite entropy and construct the homeomorphic (and even Lipschitz continuous) embedding of this system to the spatial shifts on the attractor.

Finally, we consider also the temporal evolution of the spatially chaotic structures in the attractor and prove that the spatial chaos preserves under this evolution

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Introduction

In this paper the following quasilinear parabolic boundary problem

$$\begin{cases} \partial_t u = a\Delta_x u - \lambda_0 u - f(u) + g, \quad x \in \Omega \\ u\big|_{\partial\Omega} = 0, \quad u\big|_{t=0} = u_0 \end{cases} \tag{0.1}$$

in the unbounded domain Ω (which is assumed to satisfy some natural regularity conditions formulated in §1) is considered. Here $u = (u^1, \cdots, u^k)$ is an unknown vector-valued function, f and g are given functions, $\lambda_0 > 0$ is a positive constant and a is a given $k \times k$-matrix with a positive symmetric part:

$$a + a^* > 0 \tag{0.2}$$

The longtime behavior of solutions of (0.1) is of a great interest now. It is well known that under the appropriate assumptions on the nonlinear term $f(u)$ this behavior can be described in terms of an attractor $\mathcal{A}$ of the corresponding dynamical system generated by (0.1) (see e.g. [4], [5], [25], [29]). One of the possible choices of these assumptions is the following one:

$$\begin{cases} 1. \ \ f \in C^2(\mathbb{R}^k, \mathbb{R}^k) \\ 2. \ \ f(u).u \geq -C \\ 3. \ \ f'(u) \geq -K \end{cases} \tag{0.3}$$

where $u.v$ means the standard inner product in $\mathbb{R}^k$ (see e.g. [4], [16], and [19] for the other possibilities). Note that (0.3) is fulfilled for many interesting from the physical point of view equations such as Chafee-Infante equation, Fitz-Nagumo system, generalized Ginzburg-Landau equations and other ones.

In the case where the domain Ω is bounded the global attractors for (0.1) have been constructed and studied under the various assumptions on f, a and g (see [4], [20], [29] and references therein). Particularly, the attractor's existence for (0.1) under the assumptions (0.2) and (0.3) has been proved in [34]. It is also proved there that if the nonlinearity f satisfies the additional growth restriction

$$|f(u)| \leq C(1 + |u|^p), \quad p < 1 + 4/(n-4) \tag{0.4}$$

(for $n \leq 4$ the exponent p may be arbitrarily large) then the corresponding semigroup is differentiable with respect to the initial value u_0, possesses the L^∞-bounds and the fractal dimension it's attractor is finite.

In the case where the domain Ω is unbounded (e.g. $\Omega = \mathbb{R}^n$) the situation becomes much more complicated. In this case even the choice of the appropriate phase space for (0.1) is a nontrivial problem. Indeed, the phase space $L^2(\Omega)$ (as in the case of bounded domains) seems to be not adequate because a number of natural from the physical point of view structures such as e.g. spatially periodic solutions, travelling waves, etc. are occurred to be out of the consideration. As a result the global attractor in $L^2(\Omega)$ exists for (0.1) only for very particular cases (see e.g. [5], [7], [14], [24]). That is why, following to [18], [28], [33], we will consider the equation (0.1) in the spaces

$$W_b^{l,p}(\Omega) := \{u_0 \in D'(\Omega) \, : \|u_0\|_{W_b^{l,p}} := \sup_{x_0 \in \Omega} \|u_0\|_{W^{l,p}(\Omega \cap B^1_{x_0})} < \infty\} \tag{0.5}$$

with the appropriate choice of exponents l and p (here and below $B^R_{x_0}$ means the R-ball in $\mathbb{R}^n$ centered in x_0 and $W^{l,p}(V)$ is a Sobolev space of functions whose derivatives up to the order l belong to $L^p(V)$). Roughly speaking the spaces (0.5) consist of sufficiently regular functions $u_0(x)$ which remain bounded when $|x| \to \infty$ and contain all structures mentioned above.

To the best of our knowledge the existence of the global attractor for (0.1) for the unbounded domain $\Omega = \mathbb{R}^n$ has been firstly established in [1] and [5] (for a scalar case $k = 1$ and under the great growth restrictions $p < \min\{4/n, 2/(n-2)\}$). These growth restrictions have been removed later in [17] and [24]. The case of systems ($k \geq 2$) with a scalar diffusion matrix a has been considered in [7], [14], [15], [32]. Mention also that for the particular cases of (0.1) e.g. for complex Ginzburg-Landau equations more powerful results have been obtained (see [25] and references therein).

In the present paper combining the methods of [33] and [34] we establish the existence of the global attractor for (0.1) under the assumptions (0.2) (which is much more natural from the reaction-diffusion point of view) and (0.3)-(0.4).

Theorem 1. *Let the assumptions* (0.2)–(0.4) *hold and let* $g \in L^q_b(\Omega)$ *for a some* $q \geq 2$ *such that* $q > n/2$. *Then for every* $u_0 \in \Phi_b(\Omega) := W^{2,q}_b(\Omega) \cap \{u_0\big|_{\partial\Omega} = 0\}$ *the problem* (0.1) *possesses a unique solution* $u(t) \in \Phi_b(\Omega)$ *for* $t \geq 0$ *which satisfies the following estimate:*

$$\|u(t)\|_{\Phi_b} \leq Q(\|u_0\|_{\Phi_b})e^{-\alpha t} + Q(\|g\|_{L^q_b})$$

where $\alpha > 0$ *is a positive constant and* Q *is an appropriate monotonic function which are independent of* u_0, *and consequently the solving semigroup*

$$S_t : \Phi_b(\Omega) \to \Phi_b(\Omega), \quad t \geq 0 \quad S_t u_0 := u(t) \tag{0.6}$$

is well defined for the problem (0.1).

Moreover, this semigroup possesses a bounded in $\Phi_b(\Omega)$ *and locally compact (= compact in a local topology of* $\Phi_{loc}(\Omega) := W^{2,q}_{loc}(\overline{\Omega})$*) attractor* $\mathcal{A}$.

Note that under the assumptions of Theorem 1 the Hausdorff and fractal dimension of the attractor may be infinite (and is occurred to be infinite in many interesting particular cases) (see e.g. [5], [32] or Th. 3 below) and consequently there is a problem of finding new quantitative characteristic of the attractor adopted to the infinite dimensional case. One of possible approaches to handle this problem which is suggested in [8] is to consider and estimate the Kolmogorov's ε-entropy of the infinite dimensional attractor $\mathcal{A}$.

Recall, that if K is a precompact set in a metric space M then it can be covered (due to the Hausdorff criteria) by a finite number of ε-balls for every $\varepsilon > 0$. Let $N_\varepsilon(K, M)$ be the minimal number of such balls. Then by definition the Kolmogorov's ε-entropy of K in M is the following number:

$$\mathbb{H}_\varepsilon(K, M) := \ln N_\varepsilon(K, M) \tag{0.7}$$

It is worth to emphasize that in contrast to the fractal dimension the quantity (0.7) remains finite for every $\varepsilon > 0$ and every precompact set K in M.

The ε-entropy of the infinite dimensional uniform attractors for (0.1) in the case where the domain Ω is bounded and the external force g depends explicitly on t

has been studied in [8]. The case of autonomous reaction-diffusion equations in $\mathbb{R}^n$ has been considered in [10] and [32]. The entropy for the autonomous and nonautonomous RDE in general case of the unbounded domain Ω has been considered in [15] and [33]. The entropy for damped hyperbolic equations in the unbounded domain has been investigated in [35] and [36].

It is particularly proved in [33] that in the case where the diffusion matrix a is scalar the entropy of restrictions $\mathcal{A}\big|_{\Omega\cap B^R_{x_0}}$ possesses the estimate

$$\mathbb{H}_\varepsilon(\mathcal{A}\big|_{\Omega\cap B^R_{x_0}}, \Phi_b) \le C\,\mathrm{vol}(\Omega\cap B^{R+K\ln 1/\varepsilon}_{x_0})\ln\frac{1}{\varepsilon};\quad \varepsilon\le\varepsilon_0<1 \tag{0.8}$$

where the constants C, K and ε_0 are independent of ε, R, and x_0.

In the present paper we extend this estimate to the case of general diffusion matricies a satisfying (0.2).

Theorem 2. *Let the assumptions of Theorem 1 hold. Then the entropy of the attractor $\mathcal{A}$ of* (0.1) *possesses the estimate* (0.8).

Moreover, in the case where $\Omega = \mathbb{R}^n$ and $g \equiv const$ we obtain the lower bounds for the entropy of restrictions $\mathcal{A}\big|_{B^R_{x_0}}$ under the natural assumption that (0.1) possesses at least one spatially homogeneous exponentially unstable equilibria point. Without loss of generality one may assume that $u\equiv 0$ is a such equilibria and consequently (0.1) has the following view:

$$\partial_t u = a\Delta_x u + Bu - \phi(u),\quad \phi(0)=\phi'(0)=0 \tag{0.9}$$

where the matrix $B := -f'(0)-\lambda_0$.

Theorem 3. *Let the assumptions of Theorem 1 hold and let $\Omega=\mathbb{R}^n$ and* (0.1) *has the form* (0.9)*. Assume also that*

$$\sigma(a\Delta_x + B)\cap\{z\in\mathbb{C}\ :\mathrm{Re}\, z>0\}\ne\varnothing \tag{0.10}$$

Then the entropy of the attractor possesses the following estimates:

$$\mathbb{H}_\varepsilon(\mathcal{A}\big|_{B^R_{x_0}}, \Phi_b)\ge C_1 R^n\ln\frac{1}{\varepsilon},\quad C_1>0,\quad \varepsilon\le\varepsilon_0<1 \tag{0.11}$$

Moreover, for every $\mu>0$ there is a constant $C_\mu>0$ such that

$$\mathbb{H}_\varepsilon(\mathcal{A}\big|_{B^1_{x_0}}, \Phi_b)\ge C_\mu\left(\ln\frac{1}{\varepsilon}\right)^{n+1-\mu} \tag{0.12}$$

Note that for the particular case $\Omega=\mathbb{R}^n$ (0.8) reads

$$\mathbb{H}_\varepsilon(\mathcal{A}\big|_{B^R_{x_0}}, \Phi_b)\le C_2\left(R+K\ln\frac{1}{\varepsilon}\right)^n\ln\frac{1}{\varepsilon} \tag{0.13}$$

Therefore, Theorem 3 shows that the estimate (0.8) is sharp at least in the case $\Omega=\mathbb{R}^n$. From the other side in the case where the domain Ω is bounded the estimate (0.8) yields

$$\mathbb{H}_\varepsilon(\mathcal{A},\Phi)\le C\,\mathrm{vol}(\Omega)\ln\frac{1}{\varepsilon}$$

which reflects the well-known heuristic principle that the equations of mathematical physics in bounded domains have the finite fractal dimension (and moreover indicates in a right way the dependence of this dimension on the 'size' of Ω). Thus, the estimate (0.8) may be considered as a natural generalization of this principle to the case of unbounded domains (see also [15] or [36]).

The rest part of the paper is devoted to a more comprehensive study of the spatially homogeneous case of the equation (0.1) ($\Omega = \mathbb{R}^n$, $g \equiv const$). In this case the attractor $\mathcal{A}$ possesses an additional structure, namely, it is occurred to be invariant under the group $\{T_h, h \in \mathbb{R}^n\}$ of spatial shifts:

$$(0.14) \qquad T_h : \mathcal{A} \to \mathcal{A}, \;\; T_h\mathcal{A} = \mathcal{A}, \;\; h \in \mathbb{R}^n, \;\; (T_h u_0)(x) := u_0(x+h)$$

This semigroup can be treated as a dynamical system (with multidimensional 'time' if $n > 1$) acting in the phase space $\mathcal{A}$. Thus, in order to study the spatial complexity (and spatial chaotisity) of $\mathcal{A}$ one may investigate the dynamical properties of the system (0.14).

The phenomena of spatial complexity and spatial chaotisity has been studied e.g. in [2], [6], [12] for a various particular cases of the equation (0.1). In particular, the examples which show that the topological entropy of the dynamical system (0.14) may be positive (and, moreover, that this dynamical system may contain the symbolic dynamics) has been constructed there. In the present paper we prove that under the natural assumptions the topological entropy of the dynamical system (0.14) is infinite.

Theorem 4. *Let the assumptions of Theorem 3 hold. Then the spatial dynamical system* (0.14) *has the infinite topological entropy:* $h_{sp}(\mathcal{A}) = \infty$.

Moreover, we introduce (in Section 7) a new quantitative characteristic of the dynamics – the modified topological entropy $\widehat{h}_{sp}$, which occurred to be finite and positive for the case of (0.14): $0 < \widehat{h}_{sp}(\mathcal{A}) < \infty$.

Thus, the dynamical behavior of (0.14) is occurred to be extremely chaotic. Note also that in contrast to the case of dynamical chaos, generated by ODE or by PDE in bounded domains the symbolic dynamics (Bernulli shifts, see e.g. [21]) is not an adequate model example for understanding the nature of the spatial chaotisity in (0.14) because the topological entropy of symbolic dynamics is finite. In order to overcome this difficulty a new model dynamical system which generalizes the Bernulli shifts and adopted to the case of infinite topological entropy is suggested. Namely, let $\mathbb{D}$ be a unitary disc in $\mathbb{C}$ and let $\mathcal{M} := \mathbb{D}^{\mathbb{Z}^n}$ endowed by the Tikhonov's topology. A discrete dynamical system $\mathcal{T}_h$ (with multidimensional 'time' $h \in \mathbb{Z}^n$) on $\mathcal{M}$ can be defined in a natural way:

$$(0.15) \qquad \mathcal{T}_h v(l) := v(h+l), \;\; h, l \in \mathbb{Z}^n, \;\; v \in \mathcal{M}$$

(Recall that as usual $\mathcal{M}$ is interpreted as a space of functions $v : \mathbb{Z}^n \to \mathbb{D}$).

The main result of the paper is the following theorem.

Theorem 5. *Let the assumptions of Theorem 3 hold. Then there is a positive number* $\sigma > 0$, *the closed subset* $K \subset \mathcal{A}$ *and a homeomorphism* $\tau : \mathcal{M} \to K$ *such that*

$$(0.16) \qquad T_{\sigma h}K = K \quad \textit{and} \quad T_{\sigma h}\tau(v) = \tau(\mathcal{T}_h v), \;\; \forall h \in \mathbb{Z}^n, \; v \in \mathcal{M}$$

Moreover, this homeomorphism is occurred to be Lipschitz continuous under the appropriate choice of metrics on $\mathcal{A}$ and $\mathcal{M}$ and preserves the modified topological entropy:

$$0 < \widehat{h}_{sp}(\mathcal{M}) = \widehat{h}_{sp}(K) \le \widehat{h}_{sp}(\mathcal{A}) < \infty$$

As the first elementary corollary of this construction we obtain the fact that every finite dimensional dynamics can be realized (up to a homeomorphism) by restricting the spatial dynamical system (0.14) to the appropriate closed subsets of $\mathcal{A}$.

Corollary. *Let $n = 1$ and the assumptions of Theorem 3 hold. Assume that $M \subset \mathbb{R}^N$ is an arbitrary compact set and $\psi : M \to M$ is an arbitrary homeomorphism of it. Then there is a number $\sigma' > 0$, a set $K_\psi = K_\psi(M, \psi) \subset \mathcal{A}$ and a homeomorphism $\tau' : M \to K_\psi$ such that*

$$T_{\sigma' h} K_\psi = K_\psi, \ \forall h \in \mathbb{Z} \quad \textit{and} \quad T_{\sigma'} \circ \tau' = \tau' \circ \psi \tag{0.17}$$

The result of this Corollary confirms from the other point of view that the spatial dynamics (0.14) is an extremely chaotic.

Recall now that we have also the temporal evolution operator $S_t : \mathcal{A} \to \mathcal{A}$, $t \ge 0$ generated by the equation (0.1), therefore it seems reasonable to study the temporal evolution of spatially chaotic structures in $\mathcal{A}$ (see also [11], [13]). To this end we introduce a notion of the spatial complexity for the individual point $u_0 \in \mathcal{A}$ in the following natural way:

$$\widehat{h}_{sp}(u_0) := \widehat{h}_{sp}(\mathcal{H}_{sp}(u_0)) \tag{0.18}$$

where $\mathcal{H}_{sp}(u_0) := [T_h u_0, h \in \mathbb{R}^n]_{\mathcal{A}}$ is the closure in $\mathcal{A}$ of complete orbit for u_0 with respect to the spatial shifts. Under some additional assumptions which look not very restrictive we prove that this value preserves under the temporal evolution.

Theorem 6. *Let the assumptions of Theorem 3 hold and let in addition the diffusion matrix a is normal ($aa^* = a^*a$). Then*

$$\widehat{h}_{sp}(S_t u_0) = \widehat{h}_{sp}(u_0), \quad \forall u_0 \in \mathcal{A} \tag{0.19}$$

Moreover, there are points $u_0 \in \mathcal{A}$ such that

$$0 < \widehat{h}_{sp}(u_0) < \infty$$

Thus, Theorem 6 shows that the spatial chaos preserves under the temporal evolution.

We illustrate the obtained results on the example of complex Ginzburg-Landau equation (see Example 8.1).

The paper is organized as follows.

The definitions of functional spaces which are of fundamental significance for our study the equation (0.1) and their simple properties are given in Section 1.

The various a priori estimates for the solutions of (0.1) are obtained in Section 2. Moreover, basing on these estimate we verify the existence of a solution, it's uniqueness and derive some estimates for differences of solutions which will be essentially used later.

The existence of a global attractor $\mathcal{A}$ for the system (0.1) is verified in Section 3.

The definition of Kolmogorov's ε-entropy and the standard of examples which illustrate the typical behavior of the this quantity as $\varepsilon \to 0$ for various sets in functional spaces are recalled in Section 4.

The upper bounds of the ε-entropy for the attractor $\mathcal{A}$ of the equation (0.1) are obtained in Section 5.

The further development of the method of infinite dimensional unstable manifolds for the equation (0.9) are given in Section 6. Moreover, using this method we derive the lower bounds of the Kolmogorov's entropy of the attractor and prepare a number of technical tools for studying the spatial complexity of the attractor.

This spatial complexity is investigated in Section 7 (particularly Theorems 4 and 5 are proved here). Note, that the results of this Section are essentially based on the results of Sections 5 and 6.

The temporal evolution of the spatially chaotic structures are studied in Section 8. In particular the Holder continuity of the inverse operator for S_t restricted to the attractor which is of independent interest is proved here.

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§1 Functional spaces

In this Section we introduce several classes of Sobolev spaces in unbounded domains and recall shortly some of their properties which will be essentially used below. For a detailed study of these spaces see [14], [33].

Definition 1.1. *A function $\phi \in C_{loc}(\mathbb{R}^n)$ is called a weight function with the rate of growth $\mu \geq 0$ if the condition*

$$\phi(x+y) \leq C_\phi e^{\mu|x|}\phi(y), \ \ \phi(x) > 0 \tag{1.1}$$

is satisfied for every $x, y \in \mathbb{R}^n$.

Remark 1.1. *It is not difficult to deduce from* (1.1) *that*

$$\phi(x+y) \geq C_\phi^{-1} e^{-\mu|x|}\phi(y) \tag{1.2}$$

is also satisfied for every $x, y \in \mathbb{R}^n$.

The following example of weight functions are of fundamental significance for our purposes:

$$\phi_{\varepsilon,x_0}(x) = e^{-\varepsilon|x-x_0|}, \ \ \varepsilon \in \mathbb{R}, \ \ x_0 \in \mathbb{R}^n$$

(Evidently this weight has the rate of growth $|\varepsilon|$.)

Definition 1.2. *Let $\Omega \subset \mathbb{R}^n$ be some (unbounded) domain in $\mathbb{R}^n$ and let ϕ be a weight function with the rate of growth μ. Define the space*

$$L^p_\phi(\Omega) = \left\{ u \in D'(\Omega) \ : \ \|u, \Omega, \|_{\phi,0,p} \equiv \int_\Omega \phi(x)|u(x)|^p \, dx < \infty \right\}$$

Analogously the weighted Sobolev space $W^{l,p}_\phi(\Omega)$, $l \in \mathbb{N}$ is defined as the space of distributions whose derivatives up to the order l inclusively belong to $L^p_\phi(\Omega)$.

For the simplicity of notations we will write below $W^{s,p}_{\{\varepsilon\}}$ *instead of* $W^{s,p}_{e^{-\varepsilon|x|}}$.

We define also another class of weighted Sobolev spaces

$$W^{l,p}_{b,\phi}(\Omega) = \left\{ u \in D'(\Omega) \; : \|u,\Omega\|^p_{b,\phi,l,p} = \sup_{x_0\in\Omega} \phi(x_0)\|u,\Omega\cap B^1_{x_0}\|^p_{l,p} < \infty \right\}$$

Here and below we denote by $B^R_{x_0}$ *the ball in* $\mathbb{R}^n$ *of radius* R*, centered in* x_0*, and* $\|u,V\|_{l,p}$ *means* $\|u\|_{W^{l,p}(V)}$.

We will write $W^{l,p}_b$ *instead of* $W^{l,p}_{b,1}$.

Proposition 1.1.

1. Let $u \in L^p_\phi(\Omega)$*, where* ϕ *is a weight function with the rate of growth* μ*. Then for any* $1 \le q \le \infty$ *the following estimate is valid:*

$$(1.3) \qquad \left(\int_\Omega \phi(x_0)^q \left(\int_\Omega e^{-\varepsilon|x-x_0|}|u(x)|^p\,dx\right)^q dx_0\right)^{1/q} \le C\int_\Omega \phi(x)|u(x)|^p\,dx$$

for every $\varepsilon > \mu$*, where the constant* C *depends only on* ε*,* μ *and* C_ϕ *from* (1.1) *(and independent of* Ω*).*

2. Let $u \in L^\infty_\phi(\Omega)$*. Then the following analogue of the estimate* (1.3) *is valid:*

$$(1.4) \qquad \sup_{x_0\in\Omega}\left\{\phi(x_0)\sup_{x\in\Omega}\{e^{-\varepsilon|x-x_0|}|u(x)|\}\right\} \le C\sup_{x\in\Omega}\{\phi(x)|u(x)|\}$$

The proof of this Proposition can be found in [14] or [33].

For the more detailed study of functional spaces defined above we need some regularity assumptions on the domain $\Omega \subset \mathbb{R}^n$ which are assumed to be valid throughout of the paper.

We suppose that there exists a positive number $R_0 > 0$ such that for every point $x_0 \in \Omega$ there exists a smooth domain $V_{x_0} \subset \Omega$ such that

$$(1.5) \qquad B^{R_0}_{x_0} \cap \Omega \subset V_{x_0} \subset B^{R_0+1}_{x_0} \cap \Omega$$

Moreover it is assumed also that there exists a diffeomorphism $\theta_{x_0} : B^2_0 \to B^{R_0+2}_{x_0}$ such that $\theta_{x_0}(x) = x_0 + p_{x_0}(x)$, $\theta_{x_0}(B^1_0) = V_{x_0}$ and

$$(1.6) \qquad \|p_{x_0}\|_{C^N} + \|p^{-1}_{x_0}\|_{C^N} \le K$$

where the constant K is assumed to be independent of $x_0 \in \Omega$ and N is large enough. For simplicity we suppose below that (1.5) and (1.6) hold for $R_0 = 2$.

Note that in the case when Ω is bounded the conditions (1.5) and (1.6) are equivalent to the condition: the boundary $\partial\Omega$ is a smooth manifold, but for unbounded domains the only smoothness of the boundary is not sufficient to obtain the regular structure of Ω when $|x| \to \infty$ since some uniform with respect to $x_0 \in \Omega$ smoothness conditions are required. It is the most convenient for us to formulate these conditions in the form (1.5) and (1.6).

Proposition 1.2. *Let the domain Ω satisfy the conditions* (1.5) *and* (1.6), *the weight function – the condition* (1.1) *and let R be a positive number. Then the following estimates are valid:*
(1.7)

$$C_2 \int_\Omega \phi(x)|u(x)|^p\,dx \le \int_\Omega \phi(x_0)\int_{\Omega\cap B^R_{x_0}} |u(x)|^p\,dx\,dx_0 \le C_1\int_\Omega \phi(x)|u(x)|^p\,dx$$

Proof. The proof of this Proposition is given in [14] or [33]. For the reader's convenience we recall shortly this proof.

Let us change the order of integration in the middle part of (1.7)

$$\int_\Omega \phi(x_0)\int_{\Omega\cap B^R_{x_0}} |u(x)|^p\,dx\,dx_0 = \int_\Omega |u(x)|^p\left(\int_\Omega \chi_{\Omega\cap B^R_x}(x_0)\phi(x_0)\,dx_0\right)dx \tag{1.8}$$

Here $\chi_{\Omega\cap B^R_x}$ is the characteristic function of the set $\Omega\cap B^R_x$.

It follows from the inequalities (1.1) and (1.2) that

$$C_1\phi(x) \le \inf_{x_0\in B^R_x}\phi(x_0) \le \sup_{x_0\in B^R_x}\phi(x_0) \le C_2\phi(x) \tag{1.9}$$

and the assumptions (1.5) and (1.6) imply that

$$0 < C_1 \le \mathrm{vol}(\Omega\cap B^R_x) \le C_2 \tag{1.10}$$

uniformly with respect to $x\in\Omega$.

The estimate (1.7) is an immediate corollary of the estimates (1.8)–(1.10). Proposition 1.2 is proved. □

Corollary 1.1. *Let* (1.5) *and* (1.6) *be valid. Then the equivalent norm in weighted Sobolev space $W^{l,p}_\phi(\Omega)$ is given by the following expression:*

$$\|u,\Omega\|_{\phi,l,p} = \left(\int_\Omega \phi(x_0)\|u,\Omega\cap B^R_{x_0}\|^p_{l,p}\,dx_0\right)^{1/p} \tag{1.11}$$

Particularly, the norms (1.11) *are equivalent for different $R\in\mathbb{R}_+$.*

To study the equation (0.1) we need also weighted Sobolev spaces with fractional derivatives $s\in\mathbb{R}_+$ (not only $s\in\mathbb{Z}$). For the first we recall (see [30] for details) that if V is a bounded domain the norm in the space $W^{s,p}(V)$, $s=[s]+l$, $0<l<1$, $[s]\in\mathbb{Z}_+$ can be given by the following expression

$$\|u,V\|^p_{s,p} = \|u,V\|^p_{[s],p} + \sum_{|\alpha|=[s]}\int_{x\in V}\int_{y\in V}\frac{|D^\alpha u(x)-D^\alpha u(y)|^p}{|x-y|^{n+lp}}\,dx\,dy \tag{1.12}$$

It is not difficult to prove arguing as in Proposition 1.2 and using this representation that for any bounded domain V with a sufficiently smooth boundary

$$\|u,V\|^p_{s,p} \le C_1\int_{x_0\in V}\|u,V\cap B^R_{x_0}\|^p_{s,p}\,dx_0 \le C_2\|u,V\|^p_{s,p} \tag{1.13}$$

This justifies the following definition.

Definition 1.3. *Define the space $W_\phi^{s,p}(\Omega)$ for any $s \in \mathbb{R}_+$ by the norm* (1.11).

It is not difficult to check that these norms are also equivalent for different $R > 0$. Note now that the weight functions

$$\phi_{\varepsilon,x_0}(x) = e^{-\varepsilon|x-x_0|} \tag{1.14}$$

satisfy the conditions (1.1) *uniformly* with respect to $x_0 \in \mathbb{R}^n$, consequently all estimates obtained above for the arbitrary weights will be valid for the family (1.14) with constants, independent of $x_0 \in \mathbb{R}^n$. Since these estimates are of fundamental significance for us we write it explicitly in a number of corollaries formulated below.

Corollary 1.2. *Let $u \in L^p_{\{\delta\}}(\Omega)$ for $0 < \delta < \varepsilon$. Then the following estimate holds uniformly with respect to $y \in \mathbb{R}^n$*

$$\left(\int_\Omega e^{-q\delta|x_0-y|}\left(\int_\Omega e^{-\varepsilon|x-x_0|}|u(x)|^p\,dx\right)^q dx_0\right)^{1/q} \le \\ \le C_{\varepsilon,q}\int_\Omega e^{-\delta|x-y|}|u(x)|^p\,dx \tag{1.15}$$

Moreover if $u \in L^\infty_{\{\delta\}}(\Omega)$, $\delta < \varepsilon$ then

$$\sup_{x_0\in\Omega}\left\{e^{-\delta|x_0-y|}\sup_{x\in\Omega}\{e^{-\varepsilon|x-x_0|}|u(x)|\}\right\} \le C_{\varepsilon,\delta}\sup_{x\in\Omega}\{e^{-\delta|x-y|}|u(x)|\} \tag{1.16}$$

Corollary 1.3. *Let $u \in W^{l,p}_{b,\phi}(\Omega)$ and ϕ be a weight function with the rate of growth $\mu < \varepsilon$. Then*

$$C_1\|u,\Omega\|^p_{b,\phi,l,p} \le \\ \le \sup_{x_0\in\Omega}\left\{\phi(x_0)\int_{x\in\Omega} e^{-\varepsilon|x-x_0|}\|u,\Omega\cap B^1_x\|^p_{l,p}\,dx\right\} \le C_2\|u,\Omega\|^p_{b,\phi,l,p} \tag{1.17}$$

For the proof of this corollary see [33].

We will need also the following subclass of weight functions with the exponential rate of growth.

Definition 1.4. A function $\phi \in C_{loc}(\mathbb{R}^n)$ is defined to be a weight function with the polynomial rate of growth μ if the following inequality is valid for every $x, y \in \mathbb{R}^n$

$$\phi(x+y) \le C_\phi\left((1+|y_1|^2)(1+|y_2|^2)\cdots,(1+|y_n|^2)\right)^{\mu/2}\phi(x), \quad \phi(x) > 0 \tag{1.18}$$

The following analogue of Corollary 1.3 is valid for such weights.

Corollary 1.4. *Let ϕ be a weight function with a polynomial rate of growth $\mu < N$. Then the following estimate is valid:*

$$C_1\sup_{x_0\in\Omega}\phi(x_0)u(x_0) \le \\ \le \sup_{x\in\Omega}\left\{\phi(x)\sup_{y\in\Omega}\left((1+|x_1-y_1|^2)\cdots(1+|x_n-y_n|^2)\right)^{-N/2}u(y)\right\} \le \\ \le C_2\sup_{x_0\in\Omega}\phi(x_0)u(x_0) \tag{1.19}$$

The proof of this Proposition is completely analogous to the proof of Corollary 1.3 (see e.g. [33]).

§2 The a priori estimates, existence of solutions, uniqueness.

In this Section we derive a number of a priori estimates for the solutions of the reaction-diffusion system

$$\partial_t u = a\Delta_x u - \lambda_0 u - f(u) + g, \quad x \in \Omega, \quad u\big|_{\partial\Omega} = 0, \quad u\big|_{t=0} = u_0 \tag{2.1}$$

in the unbounded domain $\Omega \subset \mathbb{R}^n$ satisfying the assumptions of the previous Section. Moreover, basing on these estimates we derive the existence of a solution $u(t)$ for (2.1) it's uniqueness and obtain some estimates for differences of solutions of (2.1) which will be used below for studying the attractor of this system.

Recall, that $u(t) = (u^1(t,x), \cdots, u^k(t,x))$ is assumed to be a vector-valued function, a is a constant $k \times k$ matrix satisfying the condition $a + a^* > 0$, $\lambda_0 > 0$, the nonlinear term $f(u)$ satisfies the assumptions

$$\begin{cases} 1. \ \ f \in C^2(\mathbb{R}^k, \mathbb{R}^k) \\ 2. \ \ f(u).u \geq -C \\ 3. \ \ f'(u) \geq -K \end{cases} \tag{2.2}$$

Moreover, we impose the additional *growth* restriction for the nonlinearity $f(u)$:

$$|f(u)| \leq C(1 + |u|^p), \tag{2.3}$$

Where the exponent p is arbitrary for $n \leq 4$ and $q < 1 + \frac{4}{n-4}$ for $n \geq 5$.

The external force g is assumed to belong to the space $L^q_b(\Omega)$ for a certain $q \geq 2$ and $q > \frac{n}{2}$ (note, that if $n \leq 3$ then the exponent $q = 2$ is admitted) and the initial data u_0 is supposed to be from the phase space $\Phi_b(\Omega) := W^{2,q}_b(\Omega) \cap \{u_0\big|_{\partial\Omega} = 0\}$.

The solution of (2.1) is defined to be a function

$$u \in L^\infty(\mathbb{R}_+, W^{2,q}_b(\Omega)) \cap C([0,\infty), L^q_b(\Omega)) \tag{2.4}$$

which satisfies the equation (2.1) in the sense of distributions.

Remark 2.1. It follows from the Sobolev's embedding theorem and from our choice of the exponent q ($q > n/2$) that the solution $u \in L^\infty(\mathbb{R}_+ \times \Omega)$, consequently, the nonlinear term in (2.1) is well-defined and belongs to L^∞. Therefore it follows from (2.4) and from the equation (2.1) that

$$\partial_t u \in L^\infty(\mathbb{R}_+, L^q_b(\Omega)) \tag{2.5}$$

Moreover, it can be shown using the standard arguments (see e.g. [33]) that

$$u \in C([0,T], W^{2,q}_{e^{-\varepsilon|x|}}(\Omega)) \cap C^1([0,T], L^q_{e^{-\varepsilon|x|}}(\Omega)) \tag{2.6}$$

for every $T > 0$ and every $\varepsilon > 0$. Note however, that in contrast to the case of bounded domains for generic $u_0 \in \Phi$ the corresponding solution $u(t)$ is *not continuous* at $t = 0$ as a function with values in $\Phi_b(\Omega)$ (see e.g. [28] for the conditions on u_0 which guarantee this continuity).

The main result of this Section is the following theorem.

Theorem 2.1. *Let the above assumptions hold and let $u(t)$ be a solution of* (2.1). *Then the following estimate is valid*

$$\|u(t)\|_{\Phi_b(\Omega)} \le Q\left(\|u(0)\|_{\Phi_b(\Omega)}\right) e^{-\alpha t} + Q\left(\|g\|_{L^q_b(\Omega)}\right) \tag{2.7}$$

where $\alpha > 0$ is a certain positive constant depending only on the equation and Q is an appropriate monotonic function which also depends only on the equation (and independent of u and u_0).

Proof. We divide the proof of this theorem in a number of lemmata.

Lemma 2.1. *Let the above assumptions hold. Then the following estimate holds for every $x_0 \in \Omega$:*

$$\begin{aligned}\|u(T), \Omega \cap B^1_{x_0}\|^2_{0,2} + \int_T^{T+1} \|u(t), \Omega \cap B^1_{x_0}\|^2_{1,2}\, dt \le \\ \le Ce^{-\alpha T}\left(e^{-\varepsilon|x-x_0|}, |u(0)|^2\right) + C\left(|g|^2, e^{-\varepsilon|x-x_0|}\right)\end{aligned} \tag{2.8}$$

where the positive constants C, α, ε are independent of x_0 and (u, v) means the inner product in $L^2(\Omega)$.

The proof of this estimate is standard and is based on multiplying the equation (2.1) by $u(t)e^{-\varepsilon|x-x_0|}$ (with $\varepsilon > 0$ small enough) integrating by parts and using the dissipativity assumption $f(u).u \ge -C$, the positiveness of a and the evident fact that

$$\|\nabla_x\left(e^{-\varepsilon|x-x_0|}\right)\| \le \varepsilon e^{-\varepsilon|x-x_0|} \tag{2.9}$$

(see e.g. [14] or [33] for details).

Lemma 2.2. *Let the above assumptions hold. Then the following estimate is valid:*

$$\begin{aligned}\|u(T), \Omega \cap B^1_{x_0}\|^2_{1,2} + \int_T^{T+1} \|u(t), \Omega \cap B^1_{x_0}\|^2_{2,2}\, dt \le \\ \le Ce^{-\alpha T}\left(e^{-\varepsilon|x-x_0|}, |u(0)|^2 + |\nabla_x u(0)|^2\right) + C\left(|g|^2, e^{-\varepsilon|x-x_0|}\right)\end{aligned} \tag{2.10}$$

where the positive constants C, α, ε are independent of x_0.

Proof. Let us multiply the equation (2.1) by the expression

$$\sum_{i=1}^n \partial_{x_i}\left(\phi_{\varepsilon,x_0}(x)\partial_{x_i} u(t)\right) := \phi_{\varepsilon,x_0}\Delta_x u(t) + \nabla_x \phi_{\varepsilon,x_0}.\nabla_x u(t) \tag{2.11}$$

where $\phi_{\varepsilon,x_0}(x) := e^{-\varepsilon|x-x_0|}$ and $\varepsilon > 0$ is small enough. Then we obtain after the standard integration by parts and using the monotonicity assumption $f'(u) \ge -K$ and the inequality (2.9) that

$$\begin{aligned}1/2\partial_t\left(\phi_{\varepsilon,x_0}, |\nabla_x u(t)|^2\right) + \lambda_0\left(\phi_{\varepsilon,x_0}, |\nabla_x u(t)|^2\right) + \mu\left(\phi_{\varepsilon,x_0}, |\Delta_x u(t)|^2\right) \le \\ \le K\left(\phi_{\varepsilon,x_0}, |\nabla_x u(t)|^2\right) + C|a|\varepsilon\left(\phi_{\varepsilon,x_0}|\Delta_x u(t)|, |\nabla_x u(t)|\right) + \\ + \left(\phi_{\varepsilon,x_0}, |g||\Delta_x u(t)| + \varepsilon|g||\nabla_x u(t)|\right)\end{aligned} \tag{2.12}$$

Estimating the last two terms in the right-hand side of (2.12) by Holder inequality we derive that

$$\begin{aligned}(2.13)\quad \partial_t\left(\phi_{\varepsilon,x_0},|\nabla_x u(t)|^2\right)+\lambda_0\left(\phi_{\varepsilon,x_0},|\nabla_x u(t)|^2\right)+\mu\left(\phi_{\varepsilon,x_0},|\Delta_x u(t)|^2\right)\le\\ \le 2K\left(\phi_{\varepsilon,x_0},|\nabla_x u(t)|^2\right)+C\left(\phi_{\varepsilon,x_0},|g|^2\right)\end{aligned}$$

Applying now the Gronwall inequality to (2.13) and using the inequality (2.8) in order to estimate the t-integral over the right-hand side of (2.13) we derive that

$$(2.14)\quad \left(\phi_{\varepsilon,x_0},|\nabla_x u(T)|^2\right)\le Ce^{-\alpha T}\left(\phi_{\varepsilon,x_0},|\nabla_x u(0)|^2+|u(0)|^2\right)+C\left(\phi_{\varepsilon,x_0},|g|^2\right)$$

The estimates (2.13) and (2.14) imply that

$$\begin{aligned}(2.15)\quad \int_T^{T+1}\left(\phi_{\varepsilon,x_0},|\Delta_x u(t)|^2\right)\,dt\le\\ \le C_1e^{-\alpha T}\left(\phi_{\varepsilon,x_0},|\nabla_x u(0)|^2+|u(0)|^2\right)+C_1\left(\phi_{\varepsilon,x_0},|g|^2\right)\end{aligned}$$

Note also, that according to our regularity assumptions on the boundary $\partial\Omega$ we have elliptic regularity for the Laplacian in Ω (see e.g. [14]):

$$(2.16)\qquad \|v\|_{W^{2,2}_{\phi_{\varepsilon,x_0}}(\Omega)}\le C\left(\|\Delta_x v\|_{L^2_{\phi_{\varepsilon,x_0}}(\Omega)}+\|v\|_{L^2_{\phi_{\varepsilon,x_0}}(\Omega)}\right)$$

The estimates (2.14)–(2.16) imply the assertion of the lemma. Lemma 2.1 is proved.

Our next task is to obtain the estimate for the $W^{2,2}_b$-norm, analogous to (2.7). To this end we introduce the following norm, depending on $\varepsilon>0$ and $x_0\in\Omega$:

$$(2.17)\qquad \|v\|^2_{D_{\varepsilon,x_0}}:=\|v\|^2_{W^{2,2}_{\phi_{\varepsilon,x_0}}(\Omega)}+\|f(v)\|^2_{L^2_{\phi_{\varepsilon,x_0}}(\Omega)}$$

Lemma 2.3. *Let the above assumptions hold and let $\varepsilon>0$ be small enough. Then the following estimate is valid for the solutions of the equation* (2.1)*:*

$$(2.18)\qquad \|u(t)\|^2_{D_{\varepsilon,x_0}}\le Ce^{2Kt}\left(\|u(0)\|^2_{D_{\varepsilon,x_0}}+1+\|g\|^2_{L^2_{\phi_{\varepsilon,x_0}}(\Omega)}\right)$$

where the constant K is the same as in (2.2) *and the constant C is independent of x_0 and ε.*

Proof. We give below only the formal deducing of the estimate (2.18) which can be easily justified using e.g. the standard difference approximations for the derivative $\partial_t u$ and the regularity (2.6).

Let us differentiate the equation (2.1) with respect to t and denote $\theta(t):=\partial_t u(t)$. Then this function satisfies the equation

$$(2.19)\qquad \partial_t\theta=a\Delta_x\theta-\lambda_0\theta-f'(u)\theta,\ \ \theta(0)=a\Delta_x u_0-f(u_0)+g,\ \ \theta\big|_{\partial\Omega}=0$$

Let us multiply this equation by $\theta(t)\phi_{\varepsilon,x_0}$ and integrate over $x\in\Omega$. Then integrating by parts and using the monotonicity assumption $f'(u)\ge -K$ and the inequality (2.9) (where ε is small enough) we derive the following estimate:

$$(2.20)\qquad \partial_t\left(\phi_{\varepsilon,x_0},|\theta(t)|^2\right)\le 2K\left(\phi_{\varepsilon,x_0},|\theta(t)|^2\right)$$

Applying the Gronwall inequality to this relation we obtain that

$$\|\partial_t u(t)\|^2_{L^2_{\phi_{\varepsilon,x_0}}(\Omega)} \le Ce^{2Kt}\left(\|u_0\|^2_{D_{\varepsilon,x_0}} + 1 + \|g\|^2_{L^2_{\phi_{\varepsilon,x_0}}(\Omega)}\right) \tag{2.21}$$

Having the estimate (2.21) for the L^2-norm of the t-derivative one can consider the parabolic equation (2.1) as an elliptic boundary value problem at a fixed point T:

$$a\Delta_x u(T) - f(u(T)) = h_u := \partial_t u(T) - g, \quad u(T)\big|_{\partial\Omega} = 0 \tag{2.22}$$

with the right-hand side h_u belonging to the space $L^2_{\phi_{\varepsilon,x_0}}(\Omega)$. Arguing as in the proof of Lemmata 2.1 and 2.2 (multiplying the equation by $u\phi_{\varepsilon,x_0}$ and by the expression (2.11) and so on) one can easily derive the estimate

$$\|u(T)\|^2_{W^{2,2}_{\phi_{\varepsilon,x_0}}(\Omega)} \le C\left(1 + \|h_u\|^2_{L^2_{\phi_{\varepsilon,x_0}}(\Omega)}\right) \tag{2.23}$$

The estimates (2.21) and (2.23) immediately imply that

$$\|u(T)\|^2_{W^{2,2}_{\phi_{\varepsilon,x_0}}(\Omega)} \le C_1 e^{2Kt}\left(\|u_0\|^2_{D_{\varepsilon,x_0}} + 1 + \|g\|^2_{L^2_{\phi_{\varepsilon,x_0}}(\Omega)}\right) \tag{2.24}$$

Thus, the $W^{2,2}$-part of the estimated (2.18) is proved. The rest part of it (the estimate of $L^2_{\phi_{\varepsilon,x_0}}$-norm of $f(u)$) is an immediate corollary of the inequalities (2.21), (2.24) and of the equation (2.1). Lemma 2.3 is proved

Applying the $\sup_{x_0\in\Omega}$ to the both sides of the inequality (2.18) and using the result of Corollary 1.3 we derive that

$$\|u(t)\|^2_{W^{2,2}_b(\Omega)} \le Ce^{2Kt}\left(\|u_0\|^2_{W^{2,2}_b(\Omega)} + \|f(u_0)\|^2_{L^2_b(\Omega)} + 1 + \|g\|^2_{L^2_b(\Omega)}\right) \tag{2.25}$$

Note, that according to our growth restrictions to f and to the Sobolev embedding theorem

$$\|f(u_0)\|_{L^2_b(\Omega)} \le Q(\|u_0\|_{W^{2,2}_b(\Omega)}) \tag{2.26}$$

for the appropriate monotonic function Q ($Q(z) := C(1 + |z|^p)$).

The inequalities (2.25) and (2.26) imply the following estimate:

$$\|u(t)\|_{W^{2,2}_b(\Omega)} \le Ce^{Kt}\left(Q(\|u_0\|_{W^{2,2}_b(\Omega)}) + \|g\|_{L^2_b(\Omega)}\right) \tag{2.27}$$

Note however that the obtained estimate of the $W^{2,2}_b$-norm *diverges* exponentially with respect to $t \to \infty$ which is not good from the attractor's point of view. In order to remove this divergence we need the following smoothing property.

Lemma 2.4. *Let the above assumptions hold. Then the following estimate is valid for any solution of the problem* (2.1)*:*

$$\|u(1)\|_{W^{2,2}_b(\Omega)} \le Q(\|u(0)\|_{W^{1,2}_b(\Omega)}) + C\|g\|_{L^2_b(\Omega)} \tag{2.28}$$

for a certain monotonic function Q.

Proof. Let us fix an arbitrary $x_0 \in \Omega$ and a sufficiently small $\varepsilon > 0$. It follows from the estimate (2.10) and the result of Theorem 1.1 that

$$\int_0^1 \|u(t)\|^2_{W^{2,2}_{\phi_{\varepsilon,x_0}}(\Omega)}\,dt \le C\left(1+\|u(0)\|^2_{W^{1,2}_{\phi_{\varepsilon,x_0}}(\Omega)}+\|g\|^2_{L^2_{\phi_{\varepsilon,x_0}}(\Omega)}\right) \tag{2.29}$$

It follows from (2.29) that there exists a point $T = T(x_0) \in [0,1]$ such that

$$\|u(T)\|^2_{W^{2,2}_{\phi_{\varepsilon,x_0}}(\Omega)} \le C\left(1+\|u(0)\|^2_{W^{1,2}_{\phi_{\varepsilon,x_0}}(\Omega)}+\|g\|^2_{L^2_{\phi_{\varepsilon,x_0}}(\Omega)}\right) \tag{2.30}$$

According to our growth restrictions to the nonlinearity $f(u)$, Sobolev embedding theorem and the result of Propositions 1.1 and 1.2 we derive that

$$\begin{aligned}
\|f(u(T))\|^2_{L^2_{\phi_{p\varepsilon,x_0}}(\Omega)} &\le C\left(1+\int_{x\in\Omega} e^{-p\varepsilon|x-x_0|}|u(T,x)|^{2p}\,dx\right) \le \\
&\le C_1\left(1+\int_{x\in\Omega} e^{-p\varepsilon|x-x_0|}\|u(T),V_x\|^{2p}_{0,2p}\,dx\right) \le \\
&\le C_2\left(1+\int_{x\in\Omega} e^{-p\varepsilon|x-x_0|}\|u(T),V_x\|^{2p}_{2,2}\,dx\right) \le \\
&\le C_3\left(1+\int_{x\in\Omega} e^{-p\varepsilon|x-x_0|}\left(\int_{y\in\Omega} e^{-\delta|y-x|}\|u(T),V_y\|^2_{2,2}\,dy\right)^p\,dx\right) \le \\
&\le C_4\left(1+\int_{x\in\Omega} e^{-\varepsilon|x-x_0|}\|u(T),V_x\|^2_{2,2}\,dx\right)^p \le C_5\left(1+\|u(T)\|^2_{W^{2,2}_{\phi_{\varepsilon,x_0}}(\Omega)}\right)^p
\end{aligned} \tag{2.31}$$

where $\delta > \varepsilon$ and V_x is the same as in the conditions (1.5) and (1.6). Here we have used also the evident formula (see e.g. [14])

$$\|v,V_x\|_{l,p} \le C_\delta \int_{y\in\Omega} e^{-\delta|x-y|}\|v,V_y\|_{l,p}\,dy \tag{2.32}$$

which holds for every $\delta > 0$.

The estimates (2.30) and (2.31) imply that

$$\|u(T)\|^2_{D_{p\varepsilon,x_0}} \le C\left(1+\|u(0)\|^2_{W^{1,2}_{\phi_{\varepsilon,x_0}}(\Omega)}+\|g\|^2_{L^2_{\phi_{\varepsilon,x_0}}(\Omega)}\right)^p \tag{2.33}$$

Applying now the estimate (2.18) with ε replaced by $p\varepsilon$ at the initial time moment $t = T$ instead of $t = 0$ we derive from (2.33) that

$$\|u(1)\|^2_{W^{2,2}_{\phi_{p\varepsilon,x_0}}(\Omega)} \le C_1\left(1+\|g\|^2_{L^2_{\phi_{\varepsilon,x_0}}(\Omega)}+\|u(0)\|^2_{W^{1,2}_{\phi_{\varepsilon,x_0}}(\Omega)}\right)^p \tag{2.34}$$

Note that all constants C_i in the previous estimates were in a fact independent of the choice of $x_0 \in \Omega$, consequently applying the $\sup_{x_0\in\Omega}$ to the both sides of (2.34) and using the result of Corollary 1.3 we derive the estimate (2.28). Lemma 2.4 is proved.

Thus, we have proved the analogue of the estimate (2.7) for $q = 2$.

Lemma 2.5. *Let the above assumptions hold. Then*

$$\|u(t)\|_{W_b^{2,2}(\Omega)} \le Q(\|u_0\|_{W_b^{2,2}(\Omega)})e^{-\alpha t} + Q(\|g\|_{L_b^2(\Omega)}) \tag{2.35}$$

for a some positive $\alpha > 0$ and a certain monotonic function Q.

Indeed, the assertion of the lemma is a simple corollary of estimates (2.10), (2.27) and (2.28)

Our task now is starting from the $W_b^{2,2}$-estimate (2.35) and using the parabolic regularity theorems to improve steps by steps this estimate to the $W_b^{2,q}$-estimate (2.7). For the first we derive the $W_b^{2-\mu,q}$-estimate for a sufficiently small positive μ.

Lemma 2.6. *Let the above assumptions hold. Then for every $\mu > 0$ the following estimate is valid:*

$$\|u(t)\|_{W_b^{2-\mu,q}(\Omega)} \le Q_\mu(\|u(0)\|_{\Phi_b(\Omega)})e^{-\alpha t} + Q_\mu(\|g\|_{L_b^q(\Omega)}) \tag{2.36}$$

where $\alpha > 0$ is a certain positive constant and Q_μ is a monotonic function (depending on μ).

Proof. Recall that we assume that the domain Ω satisfies the conditions (1.5) and (1.6) with $R_0 = 2$. Let us consider the cut-off function $\psi(x) \in C_0^\infty(\mathbb{R}^n)$ such that $\psi(x) = 1$ if $x \in B_0^1$ and $\psi(x) = 0$ if $x \notin B_0^2$. Denote $\psi_{x_0}(x) := \psi(x - x_0)$ and $v_{x_0}(t) := \psi_{x_0}u(t)$. It follows from the equation (2.1) and from the condition (1.5) that $v_{x_0}(t)$ is a solution of the following problem:

$$\begin{aligned} \partial_t v_{x_0} - a\Delta_x v_{x_0} + \lambda_0 v_{x_0} = h_{x_0}(t) := \psi_{x_0} g - 2\nabla_x\psi_{x_0}.a\nabla_x u - \\ - \Delta_x\psi_{x_0}.av_{x_0} - \psi_{x_0}f(u), \quad v_{x_0}\big|_{V_{x_0}} = 0; \quad v_{x_0}\big|_{t=0} = \psi_{x_0}u(0) \end{aligned} \tag{2.37}$$

The following standard regularity result is of fundamental significance for our proof of the lemma.

Proposition 2.1. *Let the domains V_{x_0} satisfies the assumptions* (1.5) *and* (1.6). *Then for every $1 \ge \mu > 0$, $1 < r < \infty$, and $t \in [0,1]$ the following estimate is valid for the solution v_{x_0} of the problem* (2.37)*:*

$$\|v_{x_0}(t), V_{x_0}\|_{2-\mu,r} \le C\left(\|v_{x_0}(0), V_{x_0}\|_{2-\mu,r} + \sup_{s\in[0,t]}\|h_{x_0}, V_{x_0}\|_{0,r}\right) \tag{2.38}$$

where the constant $C = C(r,\mu)$ is independent of x_0.

Moreover the following version of smoothing property is valid for every $t \in \mathbb{R}_+$:

$$\|v_{x_0}(t+1), V_{x_0}\|_{2-\mu,r} \le C_1\left(\|v_{x_0}(t), V_{x_0}\|_{1,2} + \sup_{s\in[t,t+1]}\|h_{x_0}, V_{x_0}\|_{0,r}\right) \tag{2.39}$$

where the constant C_1 is also independent of x_0.

Indeed, the estimates (2.38) and (2.39) can be easily proved using the analytic semigroups theory (see e.g. [9], [30]). Moreover the assumptions (1.5) and (1.6) imply that the constants C and C_1 are independent of x_0.

Assume now that we have already proved the estimate (2.36) with q replaced by l, $2 \le l < r$ and obtain this estimate for a larger exponent r: $q \ge r = r(l) > l$. Indeed, let $t \le 1$, then applying the $\sup_{x_0\in\Omega}$ to the both sides of (2.38) we derive that

$$\begin{aligned}(2.40)\quad \|u(t)\|_{W_b^{2-\mu,r}(\Omega)} \le C\left(\|u_0\|_{\Phi_b(\Omega)} + \|g\|_{L_b^q(\Omega)}\right) + \\ + C\sup_{s\in[0,1]}\left(\|u(s)\|_{W_b^{1,r}(\Omega)} + \|f(u(s))\|_{L_b^r(\Omega)}\right)\end{aligned}$$

Let us estimate the right-hand side of (2.40) using the $W_b^{2-\mu,l}$-norms of $u(s)$ which are assumed to be known.

Indeed, the third term into the right-hand side of (2.40) can be estimated in a such way if $r \le r_1(l) := \frac{nl}{n-l(1-\mu)}$, where $r_1 = r_1(l)$ is the Sobolev's maximal exponent of the embedding $W^{2-\mu,l} \subset W^{1,r_1}$ (as usual $r_1 = \infty$ if $n < l(1-\mu)$. Note that $r_1(l)/l > r_1(2)/2 = n/(n-2(1-\mu)) > \delta_1 > 1$.

Analogously, using the growth restriction (2.3) and Sobolev's embedding theorem $W^{2-\mu,l} \subset L^{p_\mu}$ with $p_\mu(l) := \frac{nl}{n-l(2-\mu)}$ we deduce the estimate

$$(2.41)\qquad \|f(u(s))\|_{L_b^r(\Omega)} \le C\left(1 + \|u(s)\|_{W^{2-\mu,l}(\Omega)}\right)^p$$

if $r \le r_2(l) := \frac{p_\mu(l)}{p}$. Note that according to our growth restrictions $p < \frac{n}{n-4}$ (in the case $n \le 4$ we have the embedding $W^{2,2} \subset L^r$ for every r and consequently Lemma 2.5 implies the estimate of L^r-norm of $f(u)$ for every $r < \infty$), consequently

$$(2.42)\qquad \frac{r_2(l)}{l} > \frac{r_2(2)}{2} = \frac{n}{p(n-4)}\cdot\frac{n-4}{n-4+2\mu} > \delta_2 > 1$$

if $\mu > 0$ is small enough. Let $r(l) := \min\{q, r_1(l), r_2(l)\}$. Then

$$(2.43)\qquad r(l) \ge \min\{q, \delta l\},\quad \delta := \min\{\delta_1,\delta_2\} > 1$$

if μ is small enough, and (2.40) and (2.41) imply that

$$(2.44)\qquad \|u(t)\|_{W_b^{2-\mu,r(l)}(\Omega)} \le C(1 + \|g\|_{L_b^q(\Omega)}) + C\sup_{s\in[0,1]}\|u(s)\|^p_{W_b^{2-\mu,l}(\Omega)}$$

for $t \le 1$.

Let now $t \ge 1$. Then using the estimate (2.39) instead of (2.38) and arguing as in the proof of (2.44) we derive the estimate

$$(2.45)\qquad \|u(t)\|_{W_b^{2-\mu,r(l)}(\Omega)} \le C\left(1 + \|g\|_{L_b^q(\Omega)}\right) + \sup_{s\in[t-1,t]}\|u(s)\|^p_{W_b^{2-\mu,l}(\Omega)}$$

Thus, if the analogue estimate (2.36) would be proved for some $q = l$, then the estimates (2.44) and (2.45) would imply this estimate for $q = r(l) > l$ (if $\mu > 0$ is small enough). Recall also that the estimate (2.36) for $q = 2$ is proved in Lemma 2.5. Therefore, starting with $l_0 = 2$ and iterate the estimates (2.44) and (2.45) with $l_{k+1} := r(l_k)$ we obtain finally the estimate (2.36) with $l = q$ (the finiteness of the number of iterations is guaranteed by the estimate (2.43)). Lemma 2.6 is proved.

Note that according to our assumptions on the exponent q ($q > n/2$) the embedding $W_b^{2-\mu,q} \subset C_b$ holds if $\mu > 0$ is small enough. Therefore the estimate (2.36) implies the following estimate for the C-norm of solutions of (2.1):

$$\|u(t)\|_{C_b(\Omega)} \le Q(\|u_0\|_{\Phi_b(\Omega)})e^{-\alpha t} + Q(\|g\|_{L^q_b(\Omega)}) \tag{2.46}$$

with the positive constant $\alpha > 0$ and a certain monotonic function Q.

Now we are in a position to prove that (2.36) is valid with $\mu = 0$ as well and to complete the proof of the theorem. To this end we introduce a function $\tilde{v}_{x_0} = \tilde{v}_{x_0}(x)$ as a solution of the equation

$$a\Delta_x \tilde{v}_{x_0} - \lambda_0 \tilde{v}_{x_0} + \psi_{x_0} g = 0, \quad \tilde{v}_{x_0}\big|_{\partial V_{x_0}} = 0 \tag{2.47}$$

(where ψ_{x_0} and V_{x_0} are the same as in the proof of Lemma 2.6). Then, due to the L^q-regularity theorem for the Laplacian (see e.g. [30]),

$$\|\tilde{v}_{x_0}, V_{x_0}\|_{2,q} \le C\|g, V_{x_0}\|_{0,q} \tag{2.48}$$

Moreover, due to the assumptions (1.5) and (1.6) the constant C is independent of $x_0 \in \Omega$.

Let $w_{x_0}(t) := v_{x_0}(t) - \tilde{v}_{x_0}$ where v_{x_0} is the same as in the proof of the previous lemma. Then this function evidently satisfies the equation:

$$\begin{gathered} \partial_t w_{x_0} - a\Delta_x w_{x_0} + \lambda_0 w_{x_0} = \tilde{h}_{x_0}(t) := -2\nabla_x \psi_{x_0}.a\nabla_x u(t) - \\ - \Delta_x \psi_{x_0}.au(t) - \psi_{x_0} f(u(t)), \quad w_{x_0}\big|_{\partial V_{x_0}} = 0, \quad w_{x_0}\big|_{t=0} = \psi_{x_0} u_0 - \tilde{v}_{x_0} \end{gathered} \tag{2.49}$$

The proof of the estimate (2.7) is based on (2.36) and on the following standard regularity result for the auxiliary problem (2.49).

Proposition 2.2. *Let the above assumptions hold and let $\beta > 0$ is a positive number. Then the solutions of the equation* (2.49) *satisfy the estimate*

$$\|w_{x_0}(t), V_{x_0}\|_{2,q} \le C\left(\|w_{x_0}(0), V_{x_0}\|_{2,q} + \sup_{s\in[0,1]} \|\tilde{h}_{x_0}, V_{x_0}\|_{\beta,q}\right) \tag{2.50}$$

is valid for $t \le 1$, where the constant C is independent of x_0.

Moreover, the following version of the smoothing property is valid for every $t \ge 0$ and $\mu > 0$:

$$\|w_{x_0}(t+1), V_{x_0}\|_{2+\beta-\mu,q} \le C\left(\|w_{x_0}(t), V_{x_0}\|_{1,2} + \sup_{s\in[t,t+1]} \|\tilde{h}_{x_0}(s), V_{x_0}\|_{\beta,q}\right) \tag{2.51}$$

where the constant $C = C(\beta,\mu)$ is also independent of x_0.

Indeed, the estimates (2.50) and (2.51) can be obtained using e.g. the analytic semigroups theory (see [9], [30]). The fact that the constant C is independent of x_0 is guaranteed by the regularity assumptions (1.5) and (1.6) on the domains V_{x_0}.

Note that due to the fact that $f \in C^1$ and due to the embedding $W^{2-\mu,q} \subset C$ for $\mu > 0$ is small enough we have the estimate

$$\|f(u(s))\|_{W_b^{1,q}(\Omega)} \le Q(\|u(s)\|_{W_b^{2-\mu,q}(\Omega)}) \tag{2.52}$$

for a certain monotonic function Q (depending only on f). Consequently, arguing as in the proof of Lemma 2.6 and using the estimates (2.50)–(2.52) we derive that

$$\|u(t)\|_{W_b^{2,q}(\Omega)} \le C\left(\|u_0\|_{\Phi_b(\Omega)} + \|g\|_{L_b^q(\Omega)}\right) + \sup_{s\in[0,1]} Q_1(\|u(s)\|_{W_b^{2-\mu,q}(\Omega)}) \tag{2.53}$$

is valid for $t \le 1$ and for the appropriate function Q_1 and the following smoothing property

$$\|u(t+1)\|_{W_b^{2,q}(\Omega)} \le \sup_{s\in[t,t+1]} Q_1(\|u(s)\|_{W_b^{2-\mu,q}(\Omega)}) + C\|g\|_{L_b^q(\Omega)} \tag{2.54}$$

is also valid for every $t \ge 0$. Inserting the estimate (2.36) into the right-hand side of (2.53) and (2.54) we derive after simple transformations (see e.g. [33]) the inequality (2.7). Theorem 2.1 is proved.

Remark 2.1. Arguing as in the proof of Theorem 2.1 one can deduce the following smoothing property for the solutions of (2.1)

$$\|u(1)\|_{\Phi_b(\Omega)} \le Q(\|u(0)\|_{L_b^2(\Omega)}) \tag{2.55}$$

Indeed, the smoothing property from $W_b^{1,2}(\Omega)$ to $W_b^{2,q}(\Omega)$ is in a fact proved in Lemmata 2.3–2.6. The smoothing property from L_b^2 to $W_b^{1,2}$ can be proved in a standard way (see the proof of Lemma 2.2, only instead of multiplying the equation by the expression (2.11) one should multiply it by $t(2.11)$).

As usual having the a priori estimate (2.7) one can easily verify the existence of a solution for the problem (2.1).

Theorem 2.2. *Let the above assumptions hold. Then for every $u_0 \in \Phi_b(\Omega)$ the equation* (2.1) *possesses a unique solution $u(t)$. Moreover, the following estimate holds for every two solutions $u_1(t)$ and $u_2(t)$ of the equation* (2.1)*:*

$$\|u_1(t) - u_2(t)\|_{L_b^2(\Omega)} \le Ce^{Kt}\|u_1(0) - u_2(0)\|_{L_b^2(\Omega)} \tag{2.56}$$

where the constant K is the same as in (2.2) *and constant C depends only on the equation.*

Proof. The existence of a solution of (2.1) for the case where the domain Ω is bounded can be deduced from the a priori estimate (2.7) using the Leray-Schauder fixed point principle (see e.g [23]). The existence of a solution in the unbounded domain Ω can be proved after that approximating the unbounded domain Ω by the bounded ones Ω_N and passing to the limit $N \to \infty$ (see e.g. [14] or [33] for details).

Let us prove the estimate (2.56) which immediately implies the uniqueness. Let $u_1(t)$ and $u_2(t)$ be two solutions of (2.1) and let $v(t) = u_1(t) - u_2(t)$. Then this function satisfies the equation

$$\partial_t v = a\Delta_x v - \lambda_0 v - l(t)v, \quad v\big|_{\partial\Omega} = 0, \quad v\big|_{t=0} = u_1(0) - u_2(0) \tag{2.57}$$

where $l(t) := \int_0^1 f'(su_1(t)+(1-s)u_2(t))\,ds$, $l(t) \in \mathcal{L}(\mathbb{R}^k, \mathbb{R}^k)$. Note that according to our assumptions on f, we have $l(t) \ge -K$, consequently, multiplying the equation (2.57) by $v(t)\phi_{\varepsilon,x_0}$, integrating over the $x \in \Omega$ and arguing as in the proof of Lemmata 2.1 and 2.2 we derive that

$$\|v(t)\|^2_{L^2_{\phi_{\varepsilon,x_0}}(\Omega)} + \int_t^{t+1} \|v(s)\|^2_{W^{1,2}_{\phi_{\varepsilon,x_0}}(\Omega)}\,dt \le Ce^{2Kt}\|v(0)\|^2_{L^2_{\phi_{\varepsilon,x_0}}(\Omega)} \tag{2.58}$$

Applying the operator $\sup_{x_0\in\Omega}$ to the both sides of the obtained inequality and using the result of Corollary 1.3 we obtain the inequality (2.56). Theorem 2.2 is proved.

Corollary 2.1. *Let the above assumptions hold. Then the problem* (2.1) *defines a semigroup* S_t *in the phase space* $\Phi_b(\Omega)$*:*

$$S_t : \Phi_b(\Omega) \to \Phi_b(\Omega), \quad u(t) = S_t u_0 \tag{2.59}$$

where $u(t)$ *is a solution of* (2.1) *with* $u(0) = u_0$.

Remark 2.2. The estimate (2.56) admits to extend by continuity the semigroup S_t from $\Phi_b(\Omega)$ to $L^2_b(\Omega)$. Moreover, due to the smoothing property (2.55) the semigroup $\widehat{S}_t$ thus obtained will act from $L^2_b(\Omega)$ to $\Phi_b(\Omega)$ if $t > 0$. Thus, it is possible to define a solution of the problem (2.1) for every initial data from $L^2_b(\Omega)$.

We conclude this Section by formulating some results on the smoothing property for difference of solutions of (2.1) which are of fundamental significance for our study the attractor of (2.59).

Theorem 2.3. *Let the above assumptions hold. Then for every two solutions* $u_1(t), u_2(t) \in \Phi_b$ *and for every* $\varepsilon > 0$ *the following estimate is valid:*

$$\|u_1(1) - u_2(1), \Omega \cap B^1_{x_0}\|^2_{1,2} \le C\|u_1(0) - u_2(0)\|^2_{L^2_{\phi_{\varepsilon,x_0}}(\Omega)} \tag{2.60}$$

where the constant $C = C(\|u_1\|_{\Phi_b}, \|u_2\|_{\Phi_b}, \varepsilon)$ *is independent of* $x_0 \in \Omega$. *Analogously,*

$$\|u_1(1) - u_2(1), \Omega \cap B^1_{x_0}\|^q_{2,q} \le C_1\|u_1(0) - u_2(0)\|^q_{L^2_{\phi_{\varepsilon,x_0}}(\Omega)} \tag{2.61}$$

where C_1 *is also independent of* $x_0 \in \Omega$.

Remark 2.3. Evidently the first estimate is an immediate corollary of the second one but nevertheless it is more convenient for us to formulate them separately taking in mind the further applications of them for study the entropy of the attractor.

Proof. The proof of these estimates is based on a standard analysis of the linear equation (2.57) and can be obtained in the spirit of the proof of Theorem 2.1 but essentially simpler because the equation (2.57) is linear and the coefficient $l(t)$ is smmoth enough:

$$\|l(t)\|_{W^{1,q}_b \cap C_b(\Omega)} \le Q(\|u_1(0)\|_{\Phi_b}, \|u_2(0)\|_{\Phi_b}) \tag{2.62}$$

(due to (2.7) and due to the facts that $f \in C^2$ and $W^{2,q}_b \subset C$ (see e.g. [14] or [33] for details). Indeed, in order to prove the first estimate of the theorem it is sufficient to multiply the equation (2.57) by $t\sum_{i=1}^n \partial_{x_i}(\phi_{\varepsilon,x_0}\partial_{x_i}v(t))$, integrate over $x \in \Omega$ and apply the Gronwall inequality using the estimates (2.61) and (2.58) (see the proof of Lemma 2.2). The second one can be deduced from the first one using e.g. the iteration method of improving the smoothness introduced in the proof of Lemma 2.6. Theorem 2.3 is proved.

§3 The attractor.

In this Section we prove the existence of the locally compact attractor $\mathcal{A}$ for the semigroup S_t, generated by the equation (2.1).

Note that although according to Theorem 2.1 the semigroup $S_t : \Phi_b(\Omega) \to \Phi_b(\Omega)$, generated by the equation (2.1) possesses a bounded absorbing set $\mathcal{B}$ in the phase space $\Phi_b(\Omega)$, i.e. for any other bounded subset of $B \subset \Phi_b(\Omega)$ there exists $T = T(B)$ such that

$$S_t B \subset \mathcal{B} \text{ if } t \geq T$$

(the existence of $\mathcal{B}$ is an immediate corollary of the estimate (2.7)) but nevertheless in contrast of the case of bounded domains in unbounded domains the compact attractor in $\Phi_b(\Omega)$ may not exist, e.g. the Chafee-Infante equation in $\mathbb{R}^n$ ($k = 1$, $f(u) = u^3 - \lambda u$, $\lambda > \lambda_0$) does not possess a compact attractor in the topology of $\Phi_b(\Omega)$ (see e.g. [33])

That is why (following to [17], [18], [26], [27]) we will construct below the attractor $\mathcal{A}$ of the semigroup (2.59) which attracts bounded subsets of $\Phi_b(\Omega)$ only in a local topology of the space $\Phi_{loc} = W^{2,q}_{loc}(\Omega)$ (i.e., $\mathcal{A}$ is the (Φ_b, Φ_{loc})-attractor of (2.59) in notations of [4]).

Recall that the space $\Phi_{loc}(\Omega)$ is reflexive metrizable F-space which is generated by seminorms $\|\cdot, \Omega \cap B^1_{x_0}\|_{2,q}$, $x_0 \in \Omega$.

Definition 3.1. *A set $\mathcal{A} \subset \Phi_b(\Omega)$ is defined to be the attractor of the semigroup S_t if the following assumptions hold:*

1. The set $\mathcal{A}$ is compact in $\Phi_{loc}(\Omega)$.

2. The set $\mathcal{A}$ is strictly invariant with respect to S_t, i.e.

$$S_t \mathcal{A} = \mathcal{A} \text{ for } t \geq 0$$

3. The set $\mathcal{A}$ is an attracting set for S_t in local topology, i.e. for every neighborhood $\mathcal{O}(\mathcal{A})$ of $\mathcal{A}$ in the topology of the space $\Phi_{loc}(\Omega)$ and for every bounded in uniform topology subset $B \subset \Phi_b(\Omega)$ there exists $T = T(\mathcal{O}, B)$ such that

$$S_t B \subset \mathcal{O}(\mathcal{A}) \text{ if } t \geq T$$

Recall that the first condition means that the restriction $\mathcal{A}\big|_{\Omega_1}$ is compact in the space $W^{2,q}(\Omega_1)$ for every *bounded* $\Omega_1 \subset \Omega$.

Analogously, the third condition means that for every *bounded* $\Omega_1 \subset \Omega$, every bounded B in $\Phi_b(\Omega)$ and every $W^{2,q}(\Omega_1)$-neighborhood $\mathcal{O}(\mathcal{A}\big|_{\Omega_1})$ of the restriction $\mathcal{A}\big|_{\Omega_1}$ there exists $T = T(\Omega_1, \mathcal{O}, B)$ such that

$$(S_t B)\big|_{\Omega_1} \subset \mathcal{O}(\mathcal{A}\big|_{\Omega_1}) \text{ if } t \geq T$$

Theorem 3.1. *Let the above assumptions be valid. Then the semigroup S_t, defined by* (2.59)*, possesses an attractor $\mathcal{A}$ in the sense of Definition 3.1 which has the following structure:*

$$\mathcal{A} = \mathcal{K}\big|_{t=0} \tag{3.2}$$

where we denote by $\mathcal{K}$ the set of all solutions u of (2.1)*, defined and bounded for all $t \in \mathbb{R}$ ($\sup_{t\in\mathbb{R}} \|u(t)\|_{\Phi_b(\Omega)} < \infty$).*

Proof. According to the attractor's existence theorem for abstract semigroups (see [4]), it is sufficient to verify the following conditions:

1. The semigroup S_t possesses a compact absorbing set K in Φ_{loc}-topology.

2. The operators S_t have closed graphs on K in the Φ_{loc}-topology for every fixed $t \geq 0$.

Let us verify the first condition. To this end we need the following Lemmata

Lemma 3.1. *Let the domain Ω satisfy the assumptions* (1.5) *and* (1.6). *Then for every $g \in L^q_b(\Omega)$ the problem*

$$a\Delta_x v - \lambda_0 v + g = 0, \quad v\big|_{\partial\Omega} = 0 \tag{3.3}$$

possesses a unique solution $v = v(g) \in W^{2,q}_b(\Omega)$ and the corresponding estimate

$$\|v\|_{W^{2,q}_b(\Omega)} \le C\|g\|_{L^q_b(\Omega)} \tag{3.4}$$

is valid

Indeed, the maximal regularity (3.4) follows e.g. from the estimate (2.7). The existence of a solution and it's uniqueness can be verified as in Theorem 2.2.

Lemma 3.2. *Let $u(t)$ be a solution of the equation* (2.1), *$v = v(g)$ be the solution of* (3.4) *constructed in Lemma 3.1, and $w(t) = u(t) - v$. Then there is a positive $\mu > 0$ depending only on the equation such that*

$$\|w(1)\|_{W^{2+\mu,q}_b(\Omega)} \le Q(\|u(0)\|_{\Phi_b(\Omega)}) + Q(\|g\|_{L^q_b(\Omega)}) \tag{3.5}$$

for a certain monotonic function Q.

Indeed, basing on the smoothing property (2.51) and arguing as in the end of the proof of Theorem 2.1 one can derive the estimate

$$\|w(1)\|_{W^{2+\beta,q}_b(\Omega)} \le \sup_{s\in[0,1]} Q_1(\|u(s)\|_{\Phi_b(\Omega)}) \tag{3.6}$$

for a certain monotonic function Q_1 and positive β. Inserting now the estimate (2.7) into the right-hand side of (3.6) we obtain (3.5).

The estimates (2.7) and (3.5) imply that the set

$$\begin{aligned} K := v(g) + B_R(W^{2+\beta,q}_b),\ B_R(W^{2+\beta,q}_b) :=& \\ = \{w \in W^{2+\beta,q}_b(\Omega)\ :\ \|w\|_{W^{2+\beta,q}_b} \le R\}& \end{aligned} \tag{3.7}$$

will be an absorbing set for the semigroup (2.59), generated by the equation (2.1) if R is large enough. It remains to note that the absorbing set K thus obtained is evidently compact in $\Phi_{loc}(\Omega)$. Thus, the first assumption of the abstract theorem on the attractor's existence is verified.

Let us verify the second one. To this end we need one more lemma.

Lemma 3.3. *Let $\mathcal{B}$ be a bounded set in $\Phi_b(\Omega)$ and ϕ be a positive weight function from the class introduced in Section 1 such that $\int_{\mathbb{R}^n}\phi(x)\,dx < \infty$. Then the topologies induced on $\mathcal{B}$ by the embeddings $\mathcal{B} \subset \Phi_{loc}(\Omega)$ and $\mathcal{B} \subset \Phi_\phi(\Omega) := W^{2,q}_\phi(\Omega)$ coincide.*

The assertion of the lemma is more or less evident and we leave the rigorous proof of it to a pedant reader.

Let us fix $\phi(x) = e^{-\varepsilon|x|}$ where $\varepsilon > 0$ is small enough. Then due to Lemma 3.3 in order to prove that S_t has a closed in Φ_{loc}-topology graph on K it is sufficient to prove that the convergences

$$u_0 = \Phi_\phi\text{–}\lim_{n\to\infty} u_0^n, \quad v = \Phi_\phi\text{–}\lim_{n\to\infty} S_t u_0^n \tag{3.8}$$

with $u_0^n, u_0 \in K$ imply that $v = S_t u_0$. But according to the estimate (2.58) the semigroup S_t is globally Lipschitz continuous in the L^2_ϕ-topology, consequently

$$S_t u_0 = L^2_\phi \text{--} \lim_{n\to\infty} S_t u_0^n \tag{3.9}$$

The convergences (3.8) and (3.9) imply that $v = S_t u_0$. Thus, all assumptions of the abstract theorem on the attractor's existence are verified and consequently the semigroup S_t possesses an (Φ_b, Φ_{loc})-attractor which has the structure (3.2). Theorem 3.1 is proved.

Remark 3.1. It is not difficult to prove arguing in the spirit of Section 1 that the semigroup S_t not only has a closed graph in Φ_{loc} but Lipschitz continuous and even differentiable on every Φ_b-bounded subset (see also [14]).

§4 Kolmogorov's ε-entropy: definitions and typical examples.

In this Section we recall briefly the definition of ε-entropy and give the upper and lower estimates of it when $\varepsilon \to 0$ for the typical sets in functional spaces. For the detailed study of this concept see [22], [30].

Definition 4.1. *Let $\mathbb{M}$ be a metric space and let K be precompact subset of it. For a given $\varepsilon > 0$ let $N_\varepsilon(K) = N_\varepsilon(K, \mathbb{M})$ be the minimal number of ε-balls in $\mathbb{M}$ which cover the set K (this number is evidently finite by Hausdorff criteria). By definition, Kolmogorov's ε-entropy of K in $\mathbb{M}$ is the following number:*

$$\mathbb{H}_\varepsilon(K) = \mathbb{H}_\varepsilon(K, \mathbb{M}) \equiv \ln N_\varepsilon(K) \tag{4.1}$$

Example 4.1. Let K be compact n-dimensional Lipschitz manifold in $\mathbb{M}$. Then the evident estimates imply that

$$C_1 \left(\frac{1}{\varepsilon}\right)^n \le N_\varepsilon(K) \le C_2 \left(\frac{1}{\varepsilon}\right)^n \tag{4.2}$$

and consequently

$$\mathbb{H}_\varepsilon(K) = (n + \overline{\overline{o}}(1)) \ln \frac{1}{\varepsilon} \tag{4.3}$$

when $\varepsilon \to 0$.

This example justifies the following definition.

Definition 4.2. *The fractal (box-counting) dimension of the set $K \subset\subset \mathbb{M}$ is defined to be the following number:*

$$\dim_F(K) = \dim_F(K, \mathbb{M}) = \limsup_{\varepsilon\to 0} \frac{\mathbb{H}_\varepsilon(K)}{\ln \frac{1}{\varepsilon}} \tag{4.4}$$

Note that the fractal dimension $\dim_F(K) \in [0, \infty]$ is defined for any compact set in $\mathbb{M}$ but may be not integer if K is not a manifold.

Example 4.2. Let $\mathbb{M}=[0,1]$ and let K be the ternary Cantor set in $\mathbb{M}$. Then it is not difficult to obtain that

$$C_1\left(\frac{1}{\varepsilon}\right)^d \le N_\varepsilon(K)\le C_2\left(\frac{1}{\varepsilon}\right)^d \quad , \quad d=\frac{\ln 2}{\ln 3} \tag{4.5}$$

and consequently $\dim_F(K)=d=\frac{\ln 2}{\ln 3}$.

Consider now the examples of infinite dimensional sets (i.e. $\dim_F(K)=\infty$).

The following two examples give the typical asymptotics for the entropy in the spaces of analytical functions.

Example 4.3. Let K be the set of all analytic functions f in a ball $B(R)$ of radius $R>1$ in $\mathbb{C}^n$ such that $\|f\|_{L^\infty(B(R))}\le 1$ and let $\mathbb{M}$ be the space $C(B^{Re})$, where $B^{Re}=\{z\in\mathbb{C}^n\ :\operatorname{Im} z_i=0\,,|z|\le 1\}$. Thus, K consists of all functions from $C(B^{Re})$ which can be extended holomorphically to the ball $B(R)\subset\mathbb{C}^n$ and the C-norm of this extension is not greater then one. Then

$$C_1\left(\ln\frac{1}{\varepsilon}\right)^{n+1}\le \mathbb{H}_\varepsilon(K,\mathbb{M})\le C_2\left(\ln\frac{1}{\varepsilon}\right)^{n+1} \tag{4.6}$$

For the proof of this estimate see [22].

Example 4.4. Let $\mathbb{M}$ be the same as in previous example and let K be the set of all functions f in $\mathbb{M}$ which can be extended to the entire function $\widehat{f}$ in $\mathbb{C}^n$ which satisfy the estimate

$$|\widehat{f}(z)|\le K_1e^{K_2|z|},\quad z\in\mathbb{C}^n \tag{4.7}$$

Then, as proved in [22],

$$C_1\frac{\left(\ln\frac{1}{\varepsilon}\right)^{n+1}}{\left(\ln\ln\frac{1}{\varepsilon}\right)^n}\le \mathbb{H}_\varepsilon(K)\le C_2\frac{\left(\ln\frac{1}{\varepsilon}\right)^{n+1}}{\left(\ln\ln\frac{1}{\varepsilon}\right)^n} \tag{4.8}$$

The next example gives the typical asymptotics for the entropy in the class of Sobolev spaces in bounded domains.

Example 4.5. Let Ω be smooth bounded domain in $\mathbb{R}^n$ and

$$W^{l_1,p_1}(\Omega)\subset\subset W^{l_2,p_2}(\Omega)\ ,\ 0\le l_i<\infty,\ 1<p_i<\infty,\ l_1>l_2$$

i.e., according to the embedding theorem $\frac{l_1}{n}-\frac{1}{p_1}>\frac{l_2}{n}-\frac{1}{p_2}$.

Let now $\mathbb{M}=W^{l_2,p_2}(\Omega)$ and K be the unitary ball in $W^{l_1,p_1}(\Omega)$. Then

$$C_1\left(\frac{1}{\varepsilon}\right)^{\frac{n}{l_1-l_2}}\le \mathbb{H}_\varepsilon(K)\le C_2\left(\frac{1}{\varepsilon}\right)^{\frac{n}{l_1-l_2}} \tag{4.9}$$

The proof of this estimate can be found in [30].

The following class of functions will be essentially used in the next Section in order to obtain the lower bounds of ε-entropy of attractors.

Definition 4.3. *Let us denote by $\mathbb{B}_\sigma(\mathbb{R}^n) = \mathbb{B}_\sigma(\mathbb{R}^n, \mathbb{C})$ the subspace of $L^\infty(\mathbb{R}^n, \mathbb{C})$ which consists of all functions ϕ with the Fourier transform $\widehat{\phi}$ satisfying the condition*

$$\operatorname{supp} \widehat{\phi} \subset [-\sigma, \sigma]^n \tag{4.10}$$

It is well-known that every function $\phi \in \mathbb{B}_\sigma$ can be extended to entire function $\tilde{\phi}(z) \in A(\mathbb{C}^n)$ which satisfy the estimate

$$\sup_{x\in\mathbb{R}^n} |\tilde{\phi}(x+iy)| \le C\|\phi, \mathbb{R}^n\|_{0,\infty} \cdot e^{\sigma \sum_{i=1}^n |y_i|} \tag{4.11}$$

Moreover, every function $\phi \in L^\infty$, which possesses the entire extension $\tilde{\phi}$ satisfying (4.11) belongs in fact to the space $\mathbb{B}_\sigma$.

Example 4.6. Let $K = B(0, 1, \mathbb{B}_\sigma)$, $\mathbb{M} = C(B_0^R)$. Then

$$\mathbb{H}_\varepsilon(B(0,1,\mathbb{B}_\sigma), C_b(B_0^R)) \le C(R + K \ln \frac{1}{\varepsilon})^n \ln \frac{1}{\varepsilon},\ \varepsilon \le \varepsilon_0 < 1 \tag{4.12}$$

Moreover, C and K are independent of R.

For the proof of this estimate see for instance [33]. We formulate in conclusion the lower bounds for the entropy form Example 4.6.

Proposition 4.1. *The following estimate is valid for $R \ge R_0$ and $\varepsilon < \varepsilon_0$*

$$\mathbb{H}_\varepsilon\left(B(0,1,\mathbb{B}_\sigma), C_b(B_0^R)\right) \ge CR^n \ln \frac{1}{\varepsilon} \tag{4.13}$$

where the constant C is independent of R and ε.

For the proof of (4.13) see for instance [22] or [33]. Thus, the estimate (4.12) is sharp for $R \sim \ln \frac{1}{\varepsilon}$ and $R >> \ln \frac{1}{\varepsilon}$. For the case $R << \ln \frac{1}{\varepsilon}$ we formulate only the following result (see [33]).

Proposition 4.2. *For every $\delta > 0$ there exists $C_\delta > 0$ such that*

$$\mathbb{H}_\varepsilon\left(B(0,1,\mathbb{B}_\sigma), C(B_0^1)\right) \ge C_\delta \left(\ln \frac{1}{\varepsilon}\right)^{n+1-\delta} \tag{4.14}$$

And consequently, the estimate (4.12) is sharp for the case $R << \ln \frac{1}{\varepsilon}$ also.

Remark 4.1. Instead of the spaces $\mathbb{B}_\sigma$ one can consider a slightly general class $\mathbb{B}_{\sigma,\xi}$, $\xi \in \mathbb{R}^k$ which consists of functions ϕ with Fourier transform $\widehat{\phi}$ satisfying the assumption

$$\operatorname{supp} \widehat{\phi} \subset \xi + [-\sigma, \sigma]^n \tag{4.15}$$

Note that the space $\mathbb{B}_{\sigma,\xi}$ is isomorphic to $\mathbb{B}_\sigma$ and this homeomorphism is given by multiplication on the function $e^{i\xi\cdot x}$. Consequently, the estimates (4.12) and (4.14) remain valid for the class $\mathbb{B}_{\sigma,\xi}$ as well.

We will need also the space of real parts of functions from $\mathbb{B}_{\sigma,\xi}(\mathbb{R}^n, \mathbb{C})$.

Definition 4.4. Define the space $\mathbb{B}^{Re}_{\sigma,\xi}$ by the following expression:

$$\mathbb{B}^{Re}_{\sigma,\xi}(\mathbb{R}^n, \mathbb{R}) := \{\phi \in L^\infty(\mathbb{R}^n) \, : \exists u \in \mathbb{B}_{\sigma,\xi}(\mathbb{R}^n, \mathbb{C}), \quad \phi = \operatorname{Re} u\} \tag{4.16}$$

Remark 4.2. Evidently, $\mathbb{B}^{Re}_{\sigma,\xi} \subset \mathbb{B}_{\sigma,\xi} + \mathbb{B}_{\sigma,-\xi}$. Moreover, the analogues of estimates (4.13) and (4.14) are valid for this space as well. The proof of this fact can be derived in the same way as for the case $\xi = 0$ (see e.g. [33]).

In this Section using the technique developed in [33] we obtain the upper estimates of ε-entropy for the attractor $\mathcal{A}$ of the equation (2.1). Recall that we constructed the attractor $\mathcal{A}$ which was compact only in F-space Φ_{loc} but not in the uniform topology of $\Phi_b(\Omega)$. That is why we will estimate the entropy of the restrictions $\mathcal{A}\big|_{\Omega\cap B^R_{x_0}}$ of the attractor $\mathcal{A}$ to an arbitrary ball $B^R_{x_0}$ in terms of three parameters ε, R and x_0.

Theorem 5.1. *Let the assumptions of Section 2 be valid and let*

$$\mathrm{vol}_{\Omega,x_0}(R)=\mathrm{vol}(\Omega\cap B^R_{x_0}) \tag{5.1}$$

Then for every $R\in\mathbb{R}_+$, $x_0\in\Omega$, and $\varepsilon\le\varepsilon_0<1$

$$\mathbb{H}_\varepsilon\left(\mathcal{A}\big|_{\Omega\cap B^R_{x_0}},W^{2,q}_b(\Omega\cap B^R_{x_0})\right)\le C\,\mathrm{vol}_{\Omega,x_0}(R+K\ln\frac{1}{\varepsilon})\ln\frac{1}{\varepsilon} \tag{5.2}$$

where the constants C, K and ε_0 are independent of R and $x_0\in\Omega$.

The proof of this Theorem is based on the estimates (2.60) and (2.61) with a special choice of the weight function ϕ and completely analogous to the proof of [33,Th. 8.1]. For the convenience of the reader we give below a sketch of this proof.

Define a family of weight functions with the rate of growth 1 by the following formula

$$\psi_{R,x_0}(x)=\begin{cases} e^{R-|x-x_0|} & \text{if } |x-x_0|\ge R\\ 1 & \text{if } |x-x_0|\le R\end{cases} \tag{5.3}$$

It follows from the definition of these functions that

$$\mathbb{H}_\varepsilon\left(\mathcal{A}\big|_{\Omega\cap B^R_{x_0}},W^{2,q}_b(\Omega\cap B^R_{x_0})\right)\le \mathbb{H}_\varepsilon\left(\mathcal{A},W^{2,q}_{b,\psi_{R,x_0}}(\Omega)\right) \tag{5.4}$$

Hence, instead of estimating the entropy of the restriction $\mathcal{A}\big|_{\Omega\cap B^R_{x_0}}$ it is sufficient to estimate the entropy of the attractor in weighted Sobolev spaces $W^{2,q}_{b,\psi_{R,x_0}}(\Omega)$.

Let now $u_1(t)$ and $u_2(t)$ be two solutions of the equation (2.1) which belong to the attractor $\mathcal{A}$. Then, according to the estimates (2.61)

$$\|u_1(1)-u_2(1)\|_{W^{2,q}_{b,\psi^{q/2}_{R,x_0}}(\Omega)}\le C\|u_1(0)-u_2(0)\|_{L^2_{b,\psi_{R,x_0}}(\Omega)} \tag{5.5}$$

Here the constant C is independent of $u_1,u_2\in\mathcal{A}$. (Moreover, since

$$\psi_{R,x_0}(x+y)\le e^{|x|}\psi_{R,x_0}(y)$$

then $C_{\psi_{R,x_0}}\equiv 1$ and consequently C is independent of R and x_0 also.)

Indeed, applying the operator $\sup_{z\in\Omega}\psi_{R,x_0}(z)^{q/2}$ to the both sides of (2.61) (in which x_0 is replaced by z) we obtain that

$$\|u(1)-u_2(1)\|^q_{W^{2,q}_{b,\psi^{q/2}_{R,x_0}}(\Omega)}\le$$
$$\le C\left(\sup_{z\in\Omega}\psi_{R,x_0}(z)\int_{x\in\Omega}e^{-\varepsilon|x-z|}\|u_1(0)-u_2(0),\Omega\cap B^1_x\|^2_{0,2}\,dx\right)^{q/2}$$

Applying the estimate (1.17) to the right-hand side of the previous formula we derive (5.5).

The estimate (5.5) together with the description (3.2) of the attractor $\mathcal{A}$ implies immediately that

$$\mathbb{H}_\varepsilon\left(\mathcal{A}, W^{2,q}_{b,\psi^{q/2}_{R,x_0}}(\Omega)\right) \le \mathbb{H}_{\varepsilon/(2C)}\left(\mathcal{A}, L^2_{b,\psi_{R,x_0}}(\Omega)\right) \tag{5.6}$$

The estimate (5.6) reduces our problem to estimating the entropy of the attractor in the space $L^2_{b,\psi_{R,x_0}}(\Omega)$.

The following corollary of the estimate (2.60) (which can be easily derived in the same way as (5.5)) is of fundamental significance for this estimation: let u_1 and u_2 be arbitrary two solutions of the equation (2.1) which belong to the attractor, then the following estimate is valid:

$$\|u_1(1) - u_2(1)\|_{W^{1,2}_{b,\psi_{R,x_0}}(\Omega)} \le C\|u_1(0) - u_2(0)\|_{L^2_{b,\psi_{R,x_0}}(\Omega)} \tag{5.7}$$

where the constant C depends only on the equation.

It has been proved in [33] that (5.7) implies the following recurrent estimate

Lemma 5.1[33]. *Let* (5.7) *be valid. Then*

$$\mathbb{H}_{\varepsilon/2^k}\left(\mathcal{A}, L^2_{b,\psi_{R,x_0}}\right) \le \mathbb{H}_\varepsilon\left(\mathcal{A}, L^2_{b,\psi_{R,x_0}}\right) + k\ln M_k(\varepsilon) \tag{5.8}$$

where

$$\ln M_k(\varepsilon) \le C\,\mathrm{vol}_{\Omega,x_0}(R + L\ln\frac{2^k}{\varepsilon}) \tag{5.9}$$

Moreover, the constants C and L is independent of k, R, $\varepsilon \le \varepsilon_0$ and x_0.

The estimate (5.2) is an immediate corollary of (5.8). Indeed, since $\mathcal{A}$ is bounded in Φ_b then there exists $R_0 > 0$, such that $\mathbb{H}_{R_0}(\mathcal{A}, L^2_{b,\phi_{R,x_0}}) = 0$ for every R and x_0. The estimate (5.8) implies now that

$$\mathbb{H}_{R_0/2^k}\left(\mathcal{A}, L^2_{b,\phi_{R,x_0}}\right) \le Ck\,\mathrm{vol}_{\Omega,x_0}(R + L\ln\frac{2^k}{R_0}) \tag{5.10}$$

Fixing now $k \sim \ln\frac{R_0}{\varepsilon}$ and using (5.4) and (5.6) we obtain (5.2). Theorem 5.1 is proved.

Recall now a number of standard corollaries of the estimate (5.1) (see [15], [33], and [36]).

Corollary 5.1. Since $C_b(\Omega) \subset W^{2,q}_b(\Omega)$ then

$$\mathbb{H}_\varepsilon\left(\mathcal{A}, C(\Omega \cap B^R_{x_0})\right) \le C\,\mathrm{vol}_{\Omega,x_0}(R + K\ln\frac{1}{\varepsilon})\ln\frac{1}{\varepsilon} \tag{5.11}$$

Corollary 5.2. Let $\Omega = \mathbb{R}^n$. Then $\mathrm{vol}_{\Omega,x_0}(r) = cr^n$ and consequently

$$\mathbb{H}_\varepsilon\left(\mathcal{A}, W^{2,q}_b(B^R_{x_0})\right) \le \tilde{C}\left(R + K\ln\frac{1}{\varepsilon}\right)^n \ln\frac{1}{\varepsilon} \tag{5.12}$$

Taking $R = \ln\frac{1}{\varepsilon}$ we obtain that

$$\mathbb{H}_\varepsilon\left(\mathcal{A}, W_b^{2,q}(B_{x_0}^{\ln\frac{1}{\varepsilon}})\right) \le C_1\left(\ln\frac{1}{\varepsilon}\right)^{n+1} \tag{5.13}$$

Note that the estimate (5.12) gives the same type of upper bounds for $R = 1$ and $R = \ln\frac{1}{\varepsilon}$.

Corollary 5.3. Let Ω be a bounded domain. Then Theorem 5.1 implies the estimate

$$\mathbb{H}_\varepsilon\left(\mathcal{A}, W_b^{2,q}(\Omega)\right) \le C\,\mathrm{vol}(\Omega)\ln\frac{1}{\varepsilon} \tag{5.14}$$

which reflects the well-known fact that in this case the attractor $\mathcal{A}$ has the finite fractal dimension.

Corollary 5.4. Let $\Omega = \mathbb{R}^k \times \omega^{n-k}$ be a cylindrical domain where ω is bounded. Then the estimate (5.1) gives the following bound of the ε-entropy of the attractor $\mathcal{A}$:

$$\mathbb{H}_\varepsilon\left(\mathcal{A}, W_b^{2,q}(\Omega\cap B_{x_0}^R)\right) \le C\left(R + K\ln\frac{1}{\varepsilon}\right)^k \ln\frac{1}{\varepsilon} \tag{5.15}$$

Definition 5.1 [22]. *Let $\mathcal{A} \subset \Phi_b(\Omega)$ be a compact set in the space $\Phi_{loc}(\Omega)$. Then the ε-entropy per unit volume is defined to be the following number:*

$$\overline{\mathbb{H}}_\varepsilon(\mathcal{A}) = \limsup_{R\to\infty}\frac{\mathbb{H}_\varepsilon\left(\mathcal{A}, W_b^{2,q}(\Omega\cap B_0^R)\right)}{\mathrm{vol}_{\Omega,0}(R)} \tag{5.16}$$

Corollary 5.5. The following estimate is valid:

$$\overline{\mathbb{H}}_\varepsilon(\mathcal{A}) \le C\ln\frac{1}{\varepsilon} \tag{5.17}$$

Indeed, the estimate (5.17) is an immediate corollary of the estimate (5.2) and trivial assertion

$$\lim_{R\to\infty}\frac{\mathrm{vol}_{\Omega,x_0}(R+C_1)}{\mathrm{vol}_{\Omega,x_0}(R)} = 1 \tag{5.18}$$

Definition 5.2. *Let $\widehat{h}_{sp}(\mathcal{A})$ be the following number*

$$\widehat{h}_{sp}(\mathcal{A}) = \limsup_{\varepsilon\to 0}\frac{\overline{\mathbb{H}}_\varepsilon(\mathcal{A})}{\ln\frac{1}{\varepsilon}} \tag{5.19}$$

Corollary 5.6. Let the assumptions of Theorem 6.1 hold. Then

$$\widehat{h}_{sp}(\mathcal{A}) < \infty \tag{5.20}$$

Remark 5.1. The relations between the quantity $\widehat{h}_{sp}(\mathcal{A})$ (which is called below the modified (spatial) topological entropy) and the phenomena of spatial chaotisity in the RDE in unbounded domains will be clarified in Sections 7 and 8.

§6 Infinite dimensional unstable manifolds and lower bounds of ε-entropy

In this Section we derive using the technique of infinite dimensional manifolds developed in [14], [33] the lower bounds for the entropy of the attractor $\mathcal{A}$. We restrict ourselves to consider the spatially homogeneous case $\Omega = \mathbb{R}^n$, $g \equiv 0$. Note that in this case the equation $f(z) + \lambda_0 z = 0$ always has at least one solution $z_0 = (z_0^1, \cdots, z_0^k) \in \mathbb{R}^k$ (due to the assumptions (2.2)) and consequently the equation (2.1) has at least one spatially homogeneous equilibria point $u(t) \equiv z_0$. We will obtain the lower bounds for the attractor's entropy under the additional assumption that the equation (2.1) possesses at least one exponentially unstable spatial homogeneous equilibria point $z_0 \in \mathbb{R}^k$ (without loss of generality we will assume below that $z_0 = 0$). To be more precise it is assumed that the equation (2.1) has the view

$$\partial_t u = a\Delta_x u + Bu - \tilde{f}(u) \tag{6.1}$$

where $\tilde{f} \in C^2(\mathbb{R}^k, \mathbb{R}^k)$ such that $\tilde{f}(0) = \tilde{f}'(0) = 0$, the matrix $B \in \mathcal{L}(\mathbb{R}^k, \mathbb{R}^k)$ $(B = -f'(z_0) - \lambda_0)$ and the spectrum $\sigma(\mathcal{L})$ of the linearization $\mathcal{L} := \Delta_x + B$ satisfies the assumption

$$\sigma(\mathcal{L}) \cap \{\operatorname{Re} z > 0\} \neq \varnothing \tag{6.2}$$

The main aim of this Section is to show that the assumptions (6.1) and (6.2) are sufficient for obtaining the lower bounds of the entropy of the attractor of the same type as the upper ones obtained in previous Section.

As usual we start with studying the linear nonhomogeneous problem

$$\partial_t v - \mathcal{L}v = h(t) \tag{6.3}$$

which corresponds to the linearization of (6.1) at $u \equiv 0$. To this end we need the following functional spaces.

Definition 6.1. *Let $\gamma \in \mathbb{R}$. Then the space $\mathbb{L}_\gamma(E)$, where E is a certain Banach subspace of distributions $D'(\mathbb{R}^n)$, is defined by the following expression:*

$$\mathbb{L}_\gamma(E) := \{u \in L^\infty_{loc}(\mathbb{R}_-, E) \ : \ \|u\|_{\mathbb{L}_\gamma(E)} := \sup_{t\le 0} e^{-\gamma t}\|u(t)\|_E < \infty\} \tag{6.4}$$

Lemma 6.1. *Let the exponent $\gamma > \operatorname{Re}\sigma(\mathcal{L})$. Then for every $h \in \mathbb{L}_\gamma(L^q_b(\mathbb{R}^n))$ the equation* (6.3) *possesses a backward solution $u(t)$, $t \le 0$ which is unique in the class $u \in \mathbb{L}_\gamma(W^{2-\mu,q}_b(\mathbb{R}^n))$. Thus, a linear operator*

$$\mathbb{T}_\gamma : \mathbb{L}_\gamma(L^q_b) \to \mathbb{L}_\gamma(W^{2-\mu,q}_b), \quad u(t) := (\mathbb{T}_\gamma h)(t) \tag{6.5}$$

is well defined for every $\mu > 0$. Moreover, there is a positive exponent $\varepsilon > 0$ such that

$$\|(\mathbb{T}_\gamma h)(t), B^1_{x_0}\|^q_{2-\mu,q} \le \\ \le C_\mu \sup_{s\in(-\infty,t]} e^{(\gamma+\varepsilon)(t-s)} \left(\sup_{x\in\mathbb{R}^n} e^{-\varepsilon|x-x_0|}\|h(s), B^1_x\|^q_{0,q}\right) \tag{6.6}$$

where the constant C_μ is independent of x_0 and t.

Proof. Note that due to the smoothing property for solutions of the linear equation (6.3) (see Propositions 2.1, 2.2 and Theorem 2.3) it is sufficient to deduce the estimate (6.6) only for $W^{1,2}$-norm in the left-hand side (instead of $W^{2-\mu,q}$-norm). Note also that without loss of generality we may assume that $\gamma = 0$.

Let us consider for the first the case where $h \in \mathbb{L}_0(L^2(\mathbb{R}^n))$ (the general case will be reduced below to this one). It is well known (see e.g. [30]) that the operator $\mathcal{L}$ generates an analytic semigroup in $L^2(\mathbb{R}^n)$ and consequently, due to the spectral mapping theorem, $\sigma(e^{\mathcal{L}})\backslash\{0\} = e^{\sigma(\mathcal{L})}$ (see e.g. [9]). Note also that according to our assumption $\operatorname{Re}\sigma(\mathcal{L}) < 0$ ($\gamma = 0!$) therefore there is a positive $\nu > 0$ such that $\operatorname{Re}\sigma(\mathcal{L}) < -2\nu$. Thus, the spectral radius of the exponent $e^{\mathcal{L}}$ satisfies the inequality

$$r(e^{\mathcal{L}}) \le e^{-2\nu} < 1 \tag{6.7}$$

and consequently, the Duhamel formula

$$v(t) := \int_{-\infty}^{t} e^{\mathcal{L}(t-s)} h(s)\, ds \tag{6.8}$$

defines a solution $v \in \mathbb{L}_0(L^2(\mathbb{R}^n))$ which satisfies the estimate

$$\|v(t)\|_{L^2(\mathbb{R}^n)} \le C \sup_{s\in(-\infty,t]} e^{-\nu(t-s)}\|h(s)\|_{L^2(\mathbb{R}^n)}, \quad t \le 0 \tag{6.9}$$

Moreover, this solution is unique in the class $\mathbb{L}_0(L^2)$.

The estimate (6.9) together with a standard $(L^2, W^{1,2})$-smoothing property for the solutions of (6.3) yield

$$\|v(t)\|_{1,2} \le C_1 \sup_{s\in(-\infty,t]} e^{-\nu(t-s)}\|h(s)\|_{0,2} \tag{6.10}$$

It is convenient for us to write the last estimate in the following equivalent form:

$$\sup_{s\in(-\infty,t]} e^{-\nu(t-s)}\|v(s)\|_{1,2} \le C_2 \sup_{s\in(-\infty,t]} e^{-\nu(t-s)}\|h(s)\|_{0,2} \tag{6.11}$$

In order to reduce the general case $h \in \mathbb{L}_0(L^2_b)$ to the one considered above we fix an arbitrary $x_0 \in \mathbb{R}^n$ and introduce a new unknown function $w_{x_0}(t) := v(t)\tilde\phi_{\varepsilon,x_0}$, where $\tilde\phi_{\varepsilon,x_0}(x) := e^{-\varepsilon(1+|x-x_0|^2)^{1/2}}$ and $\varepsilon > 0$ is a small parameter which will be specified below. Note that the weight functions $\tilde\phi_{\varepsilon,x_0}$ are equivalent to ϕ_{ε,x_0} but smooth and satisfy the following conditions

$$|\nabla_x\tilde\phi_{\varepsilon,x_0}| \le C\varepsilon\tilde\phi_{\varepsilon,x_0}, \quad |D^2\tilde\phi_{\varepsilon,x_0}| \le C\varepsilon^2\tilde\phi_{\varepsilon,x_0} \tag{6.12}$$

It is not difficult to verify that the function w_{x_0} satisfies the equation

$$\partial_t w_{x_0} - \mathcal{L}w_{x_0} = \tilde\phi_{\varepsilon,x_0}h + K_1(x)w_{x_0} + K_2(x)\nabla_x w_{x_0} := h_{x_0}(t) \tag{6.13}$$

Moreover, the estimates (6.12) imply that $|K_i(x)| \le C_2\varepsilon$.

Evidently $h_{x_0} \in \mathbb{L}_0(L^2)$, consequently the estimate (6.11) yields

$$
\begin{aligned}
(6.14) \qquad \sup_{s\in(-\infty,t]} e^{-\nu(t-s)}\|w_{x_0}(s)\|_{1,2} &\le \sup_{s\in(-\infty,t]} e^{-\nu(t-s)}\|h_{x_0}(s)\|_{0,2} \le \\
&\le C_3 \sup_{s\in(-\infty,t]} e^{-\nu(t-s)}\|\phi_{\varepsilon,x_0}h(s)\|_{0,2} + C_3\varepsilon \sup_{s\in(-\infty,t]} e^{-\nu(t-s)}\|w_{x_0}(s)\|_{1,2}
\end{aligned}
$$

Fixing in (6.14) $\varepsilon > 0$ small enough we derive that

$$
\begin{aligned}
\|v(t), B^1_{x_0}\|_{1,2} &\le C \sup_{s\in(-\infty,t]} e^{-\nu(t-s)}\|\phi_{\varepsilon,x_0}v(s)\|_{1,2} \le \\
&\le C_1 \sup_{s\in(-\infty,t]} e^{-\nu(t-s)} \sup_{x\in\mathbb{R}^n}\left\{\phi_{\varepsilon/2,x_0}(x)\|v(s), B^1_x\|_{1,2}\right\}
\end{aligned}
$$

The estimate (6.6) is proved. Applying the operator $\sup_{t\in\mathbb{R}_-} e^{-\gamma t}\sup_{x_0\in\mathbb{R}^n}$ to the both sides of the inequality (6.6) we derive, using (1.4) that

$$
(6.15) \qquad \|v\|_{\mathbb{L}_\gamma(W_b^{2-\mu,q})} \le C_5\|h\|_{\mathbb{L}_\gamma(L^q_b)}
$$

Lemma 6.1 is proved.

Corollary 6.1. *Let the assumptions of Lemma 6.1 hold and let ϕ be a weight function which satisfies* (1.1) *with a sufficiently small rate of growth. Then the operator $\mathbb{T}_\gamma$ constructed in Lemma 6.1 is bounded as the operator from $\mathbb{L}_\gamma(L^q_{b,\phi})$ to $\mathbb{L}_\gamma(W_{b,\phi}^{2-\mu,q})$*

Indeed, this assertion is an immediate corollary of (6.6) and (1.4).

Let us study now the homogeneous problem (6.3) (the case $h \equiv 0$).

Lemma 6.2. *Let the spectrum of $\mathcal{L}$ satisfy the assumption* (6.2)*. Then there exist $\gamma > 0$, $\sigma > 0$, $\xi_0 \in \mathbb{R}^k$, $e \in \mathbb{R}^k$ and the operator $\mathcal{P}_\gamma : \mathbb{B}_{\sigma,\xi_0} \to \mathbb{L}_\gamma(W_b^{2,q}(\mathbb{R}^n, \mathbb{C}^k))$ (where the space $\mathbb{B}_{\sigma,\xi_0} := \mathbb{B}_{\sigma,\xi_0}(\mathbb{R}^n, \mathbb{C})$ is defined by* (4.15)*) such that*

1. For every $u_0 \in \mathbb{B}_{\sigma,\xi_0}(\mathbb{R}^n)$ the function $v \in \mathbb{L}_\gamma(W_b^{2,q}(\mathbb{R}^n))$ defined by $v(t) := \mathcal{P}_\gamma(u_0)(t)$, $t \le 0$ is a solution of (6.3) *with $h \equiv 0$.*

2. $2\gamma > \operatorname{Re}\sigma(\mathcal{L})$.

3. Let $S_\gamma(u_0) := \mathcal{P}_\gamma(u_0)(0)$ and let $\Pi_e z := \frac{z.e}{|e|^2}$ is the orthogonal projection to the vector e, then $\Pi_e S_\gamma(u_0) = u_0$ for every $u_0 \in \mathbb{B}_{\sigma,\xi_0}$.

4. For every $N \in \mathbb{R}_+$ and $u_0 \in \mathbb{B}_{\sigma,\xi_0}$ the following estimate holds

$$
(6.16) \qquad \|\mathcal{P}_\gamma(u_0)(t), B^1_{x_0}\|_{2,q} \le C_N e^{\gamma t}\sup_{x\in\mathbb{R}^n}\left\{\frac{1}{(1+|x-x_0|^{2N})^{1/2}}\|u_0, B^1_x\|_{0,\infty}\right\}
$$

Moreover, the constant C_N is independent of $x_0 \in \mathbb{R}^n$.

Proof. Applying the x-Fourier transform to homogeneous equation (6.3) we will have the equation

$$
(6.17) \qquad \partial_t\widehat{v}(t) - \widehat{\mathcal{L}}(\xi)\widehat{v}(t) = 0
$$

where $\widehat{\mathcal{L}}(\xi) := -a|\xi|^2 + B$. Note, that the assumption (6.2) implies that there is a point $\xi_0 \in \mathbb{R}^k$ and $\widehat{\lambda}_0 \in \sigma(\widehat{\mathcal{L}}(\xi_0))$ such that $\operatorname{Re}\widehat{\lambda}_0 > 0$. Moreover, without loss of generality we may assume that $\operatorname{Re}\sigma(\widehat{\mathcal{L}}(\xi)) < \widehat{\lambda}_0 + \varepsilon$ for every $\xi \in \mathbb{R}^k$, where $\varepsilon > 0$ is small enough to satisfy $\varepsilon < \operatorname{Re}\widehat{\lambda}_0/3$.

Let us denote by $\widehat{\lambda}(\xi)$ the spectrum of $\sigma(\widehat{\mathcal{L}}(\xi))$. Then (since the pencil $\widehat{\mathcal{L}}(\xi)$ is polynomial with respect to ξ) $\widehat{\lambda}(\xi)$ is an analytic function with respect to ξ on the corresponding k-sheeted Riemann surface. Moreover, without loss of generality we may assume also that $\xi_0 \neq 0$ and is not a branch point for this function. Denote by $\widehat{\lambda}_0(\xi)$ the analytic branch of $\widehat{\lambda}(\xi)$ in the neighborhood of ξ_0 such that $\widehat{\lambda}_0(\xi_0) = \widehat{\lambda}_0$.

Thus, we have proved that there exists a neighborhood $B^{r_0}_{\xi_0}$ of ξ_0 and smooth functions $\widehat{\lambda}_0 : B^{r_0}_{\xi_0} \to \mathbb{C}$ and $e_0 : B^{r_0}_{\xi_0} \to \mathbb{C}^k$ such that

$$\widehat{\mathcal{L}}(\xi)e_0(\xi) = \widehat{\lambda}_0(\xi)e_0(\xi), \quad e_0(\xi) \neq 0 \tag{6.18}$$

Evidently, we may fix $r_0 > 0$ in such a way that $\operatorname{Re}\widehat{\lambda}_0(\xi) > \operatorname{Re}\lambda_0 - \varepsilon$ for every $\xi \in B^{r_0}_{\xi_0}$ and $r_0 < |\xi_0|$. Moreover, since $e_0(\xi_0) \neq 0$ then either $\operatorname{Re} e_0(\xi_0) \neq 0$ or $\operatorname{Im} e_0(\xi_0) \neq 0$. Define $e := \operatorname{Re} e_0(\xi_0)$ if $\operatorname{Re} e_0(\xi_0) \neq 0$ and $e := \operatorname{Im} e_0(\xi_0)$ otherwise. Then it it is possible to normalize the eigenvector $e_0(\xi_0)$ in such a way that

$$\Pi_e e_0(\xi) \equiv 1, \quad \text{for every } \xi \in B^{r_0}_{\xi_0} \tag{6.19}$$

(decreasing the radius r_0 if necessary).

Let us fix now the exponent $\sigma > 0$ and the corresponding space $\mathbb{B}_{\sigma,\xi_0}$ in such a way that $\operatorname{supp}\widehat{\phi} \subset B^{r_0/2}_{\xi_0}$ for every $\phi \in \mathbb{B}_{\sigma,\xi_0}$ and define the solution of (6.3) by the expression

$$\widehat{v}(t,\xi) := e^{\widehat{\lambda}_0(\xi)t}\widehat{\phi}(\xi)e_0(\xi) \tag{6.20}$$

We claim that the operator $\mathcal{P}_\gamma : \phi \to v$, where $\gamma = \operatorname{Re}\widehat{\lambda}_0 - \varepsilon$, defined by (6.20) satisfies all assumptions of the Lemma.

Indeed, define a cut-off function $\psi \in C_0^\infty(\mathbb{R}^n)$ such that $\psi(\xi) \equiv 1$ if $\xi \in B^{r_0/2}_{\xi_0}$ and $\psi(\xi) = 0$ if $\xi \notin B^{r_0}_{\xi_0}$. Then the formula (6.20) can be rewritten in the following equivalent form:

$$\widehat{v}(t,\xi) = e^{\gamma t}\Psi(t,\xi)\widehat{\phi}(\xi), \quad \xi \in \mathbb{R}^k \tag{6.21}$$

where $\Psi(t,\xi) := e^{(\widehat{\lambda}_0(\xi) - \widehat{\lambda}_0 + \varepsilon)t}\psi(\xi)e_0(\xi)$. Moreover, it is not difficult to verify that due to our construction of functions $\psi, \widehat{\lambda}_0$ and e_0

$$\int_{\mathbb{R}^n} |D^N\Psi(t,\xi)|^2\, d\xi \le C_N \tag{6.22}$$

uniformly with respect to $t \in \mathbb{R}_-$. Thus, the operator $\mathcal{P}_\gamma$ can be represented as a convolution operator

$$\mathcal{P}_\gamma(u_0)(t) = e^{\gamma t}\left(F^{-1}_\xi \Psi(t,\xi)\right) * u_0, \quad u_0 \in \mathbb{B}_{\sigma,\xi_0} \tag{6.23}$$

Moreover, it follows from (6.22) that the convolution's kernel $K(t,x)$ in (6.23) satisfies the estimate

$$|K(t,x)| := |(F^{-1}_\xi \Psi(t,\xi)(x)| \le C_N \frac{1}{(1+|x|^{2N})^{1/2}} \tag{6.24}$$

for every $N \in \mathbb{R}_+$ and consequently

$$|v(t,x_0)| \le \tilde C_N e^{\gamma t} \sup_{x\in\mathbb{R}^k} \frac{\|u_0, B^1_x\|_{0,\infty}}{(1+|x-x_0|^{2N})^{1/2}} \tag{6.25}$$

The estimate (6.16) is an immediate corollary of (6.25) and the smoothing property for the linear equation (6.3). (Note that this estimate implies particularly that the operator $\mathcal{P}_\gamma$ is really a bounded operator from B_{σ,ξ_0} to $\mathbb{L}_\gamma(W^{2,q}_b)$). The rest properties of $\mathcal{P}_\gamma$ announced in Lemma 6.2 are evident. Indeed, the fact that for every $u_0 \in \mathbb{B}_{\sigma,\xi_0}$ $v := \mathcal{P}_\gamma u_0$ is a solution of (6.3) follows from the representation (6.20). The second assertion is a corollary of our choice of the exponent ε ($2\gamma = 2(\widehat\lambda_0 - \varepsilon) > \widehat\lambda_0 + \varepsilon > \operatorname{Re}\sigma(\mathcal{L})$, because $\varepsilon < \widehat\lambda_0/3$) and the third one is a corollary of the normalization (6.19). Lemma 6.2 is proved.

Corollary 6.2. *Let the assumptions of Lemma 6.2 hold. Then for every weight function ϕ with a polynomial rate of growth (see* (1.18)*) the following estimate is valid:*

$$\|\mathcal{P}_\gamma(u_0)(t)\|_{W^{2,q}_{b,\phi}(\mathbb{R}^n)} \le Ce^{\gamma t}\|u_0\|_{L^\infty_{b,\phi^{1/q}}(\mathbb{R}^n)}, \quad u_0 \in \mathbb{B}_{\sigma,\xi_0} \tag{6.26}$$

where the constant C is independent of the concrete choice of the weight ϕ satisfying (1.18).

Indeed the assertion of the lemma is an immediate corollary of (6.16) and (1.19).

Recall we have constructed the *complex* valued solution $\mathcal{P}_\gamma(u_0)$ of the equation (6.3) but we need in the following only the *real* valued solutions of this equation. Since the operator $\mathcal{L}$ has real coefficients then $\operatorname{Re}\mathcal{P}_\gamma(u_0)$ is the appropriate real-valued solution. Moreover, the assertions of Lemma 6.2 remain valid for this operator except of p. 3, which should be replaced by

$$\Pi_e S_\gamma(u_0) = \operatorname{Re} u_0, \quad \text{for every } u_0 \in \mathbb{B}_{\sigma,\xi_0} \tag{6.27}$$

Note however, that $\operatorname{Re} u_0, u_0 \in \mathbb{B}_{\sigma,\xi_0}$ if and only if $u_0 \equiv 0$ (due to the fact that by definition $\operatorname{supp}\widehat u_0 \subset B^{r_0}_{\xi_0}$ and $r_0 < |\xi_0|$. Moreover, the following is true.

Proposition 6.1. *Let $\sqrt{n}\sigma < |\xi_0|$. Then a function $u_0 \in \mathbb{B}_{\sigma,\xi_0}(\mathbb{R},\mathbb{C})$ is uniquely determined by it's real part* $\operatorname{Re} u_0$. *Moreover, for every $N \in \mathbb{R}_+$ the following estimate is valid:*

$$|u_0(x_0)| \le C_N \sup_{x\in\mathbb{R}^n} \frac{\|\operatorname{Re} u_0, B^1_x\|_{0,\infty}}{(1+|x-x_0|^{2N})^{1/2}} \tag{6.28}$$

where the constant C_N is independent of $x_0 \in \mathbb{R}^n$ and consequently the spaces $\mathbb{B}^{Re}_{\sigma,\xi_0}$ and $\mathbb{B}_{\sigma,\xi_0}$ are isomorphic. We denote this isomorphism by $\mathcal{R}$.

Proof. Indeed, since $\overline{u}_0 \in \mathbb{B}_{\sigma,-\xi_0}$ and $\sqrt{n}\sigma < |\xi_0|$ then

$$\operatorname{supp}\widehat u_0 \cap \operatorname{supp}\widehat{\overline{u}_0} = \varnothing$$

Let $\psi(\xi) \in C_0^\infty(\mathbb{R}^n)$ be a cut-off function, such that $\psi(\xi) \equiv 1$ if $\xi \in \xi_0 + [-\sigma,\sigma]^n$ and $\psi(\xi) \equiv 0$ if $\xi \in -\xi_0 + [-\sigma,\sigma]^n$ and let $K(x) := F^{-1}_{\xi\to x}\psi$. Then

$$u_0 = 2K * \operatorname{Re} u_0 \tag{6.29}$$

and $|K(x)| \le C_N(1+|x|^{2N})^{-1/2}$. The estimate (6.28) is an immediate corollary of (6.29). Proposition 6.1 is proved.

We will write below $\mathcal{P}_\gamma$ instead $\operatorname{Re}\mathcal{P}_\gamma$ and S_γ instead of $\operatorname{Re} S_\gamma$ where it will not lead to misunderstanding.

Corollary 6.3. *Let ϕ be a weight function with the polynomial rate of growth (see* (1.18)*) and let the assumptions of Lemma 6.2 hold. Then the following estimate is valid:*

$$\|u_0\|_{L^\infty_{b,\phi}} \le C\|S_\gamma u_0\|_{L^\infty_{b,\phi}}, \quad u_0 \in \mathbb{B}_{\sigma,\xi_0} \tag{6.30}$$

where $S_\gamma u_0 := (\operatorname{Re}\mathcal{P}_\gamma u_0)(0)$.

Indeed, the assertion of this corollary follows from (6.27) (6.28) and (1.19).

We are in a position now to formulate the main technical result of this Section.

Theorem 6.1. *Let the assumptions of Theorem 3.1 be valid and let in addition the equation* (2.1) *can be represented in the form* (6.1) *with the exponentially unstable linear part (the assumption* (6.2) *is also assumed to be satisfied). Then there exists* $r > 0$ *and a* C^1*-map*

$$\mathcal{U}_0 : B(0, r, \mathbb{B}_{\sigma,\xi_0}(\mathbb{R}^n, \mathbb{C})) \to \mathcal{A} \tag{6.31}$$

where $B(0, r, \mathbb{B}_{\sigma,\xi_0})$ *is a* r*-ball in the space* $\mathbb{B}_{\sigma,\xi_0}$ *centered in* 0 *and the constants* σ, ξ_0 *are the same as in Lemma 6.2, and for every* $u_0 \in B(0, r, \mathbb{B}_{\sigma,\xi_0})$ *the following estimate is valid*

$$\|\mathcal{U}_0(u_0) - S_\gamma(u_0)\|_{\Phi_b(\mathbb{R}^n)} \le C\|u_0\|^2_{L^\infty_b(\mathbb{R}^n)} \tag{6.32}$$

Moreover, this map is a Lipschitz continuous embedding in the local topology in the following sense: for every $N \in \mathbb{R}_+$ *and every* $x_0 \in \mathbb{R}^k$ *we have the estimates*

$$\begin{cases} \|\mathcal{U}_0(u_1) - \mathcal{U}_0(u_2), B^1_{x_0}\|_{2,q} \le C_N \sup_{x\in\Omega} \frac{\|u_1 - u_2, B^1_x\|_{0,\infty}}{(1+|x-x_0|^{2N})^{1/2}} \\ \|u_1 - u_2, B^1_{x_0}\|_{0,\infty} \le C_N \sup_{x\in\Omega} \frac{\|\mathcal{U}_0(u_1) - \mathcal{U}_0(u_2), B^1_x\|_{2,q}}{(1+|x-x_0|^{2N})^{1/2}} \end{cases} \tag{6.33}$$

which are valid for every $u_1, u_2 \in B(0, r, \mathbb{B}_{\sigma,\xi_0})$.

Proof. The proof of this theorem is based on the implicit function theorem and on the following lemma.

Lemma 6.3. *Let* $f \in C^2$ *satisfies* $f(0) = f'_u(0) = 0$ *and let the exponent* $\mu > 0$ *be fixed in such a way that the embedding* $W^{2-\mu,q} \subset C$ *holds. Then the Nemitskij operator* $Fu = f(u)$ *belongs to the space* $C^1(\mathbb{L}_\gamma(W^{2-\mu,q}_b), \mathbb{L}_{2\gamma}(L^q_b))$.

The assertion of this lemma can be verified in a direct way (see [36], for example).

Now we are going to find the backward solutions of the problem (6.1) near $z_0 = 0$ equilibria point using the implicit function theorem. To this end we rewrite this equation in the form

$$\partial_t u - \mathcal{L}u = -\tilde{f}(u), \quad t \le 0$$

Let us fix γ such as in Lemma 6.2, μ as in Lemma 6.3 and consider the equation

$$u + \mathbb{T}_{2\gamma}\tilde{f}(u) = \mathcal{P}_\gamma u_0, \quad u \in \mathbb{L}_\gamma(W^{2-\mu,q}_b) \tag{6.34}$$

where $u_0 \in \mathbb{B}_{\sigma,\xi_0}$ and σ satisfies the conditions of Lemma 6.2. Note that every solution of (6.34) is simultaneously a solution of the equation (6.1) hence it is sufficient to solve (6.34) in $\mathbb{L}_\gamma(W^{2-\mu,q}_b)$.

To this end we introduce a function $\mathcal{F} : \mathbb{L}_\gamma(W_b^{2-\mu,q}) \times \mathbb{B}_{\sigma,\xi_0} \to \mathbb{L}_\gamma(W_b^{2-\mu,q})$ by formula

$$\mathcal{F}(u, u_0) = u + \mathbb{T}_{2\gamma}\tilde{f}(u) - \mathcal{P}_\gamma u_0$$

It follows from Lemmata 6.1, 6.2 and 6.3 that the function $\mathcal{F}$ belongs to the class $C^1(\mathbb{L}_\gamma(W_b^{2-\mu,q}) \times \mathbb{B}_{\sigma,\xi_0}, \mathbb{L}_\gamma(W_b^{2-\mu,q}))$ and $D_u\mathcal{F}(0,0) = Id$. Hence due to the implicit function theorem (see [31] for instance) there exists a neighborhood $B(0, r, \mathbb{B}_{\sigma,\xi_0})$ and a C^1-function

$$\mathcal{U} : B(0, r, \mathbb{B}_{\sigma,\xi_0}) \to \mathbb{L}_\gamma(W_b^{2-\mu,q})$$

such that $\mathcal{F}(\mathcal{U}(u_0), u_0) \equiv 0$ and consequently $\mathcal{U}(u_0)(t)$ is a backward solution of the problem (6.1). The equation (6.34) and Lemmata 6.1–6.3 imply now that

$$(6.35)\quad \|\mathcal{U}(u_0) - \mathcal{P}_\gamma u_0\|_{\mathbb{L}_{2\gamma}(W_b^{2-\mu,q})} \le C\|\tilde{f}(\mathcal{U}(u_0))\|_{\mathbb{L}_{2\gamma}(L_b^q)} \le \\ \le C_1\|\mathcal{U}(u_0)\|^2_{\mathbb{L}_\gamma(W_b^{2-\mu,q})} \le C_2\|u_0\|^2_{\mathbb{B}_{\sigma,\xi_0}}$$

Recall that the function $u(t) := \mathcal{U}(u_0)(t)$ satisfies the equation (6.1). Consequently, due to the smoothing property for the nonlinear equation (6.1) (see Proposition 2.2 and the end of the proof of Theorem 2.1) and due to the fact that $\tilde{f}(0) = 0$ we derive that

$$\|u(t+1)\|_{\Phi_b} \le Q(\|u(t)\|_{W_b^{2-\mu,q}})\|u(t)\|_{W_b^{2-\mu,q}}$$

and therefore

$$(6.36)\quad \|\mathcal{U}(u_0)\|_{\mathbb{L}_\gamma(\Phi_b)} \le Q(\|\mathcal{U}(u_0)\|_{\mathbb{L}_0(W_b^{2-\mu,q}(\Omega))})\|\mathcal{U}(u_0)\|_{\mathbb{L}_\gamma(W_b^{2-\mu,q})} \le C\|u_0\|_{\mathbb{B}_{\sigma,\xi_0}}$$

for every $u_0 \in B(0, r, \mathbb{B}_{\sigma,\xi_0})$. Analogously, the function $w(t) := \mathcal{U}(u_0)(t) - \mathcal{P}_\gamma u_0$ satisfies the equation

$$\partial_t w(t) - a\Delta_x w(t) - Bw(t) = -\tilde{f}(u(t))$$

Applying the smoothing property to this equation and using (6.36) and the fact that $\tilde{f}(0) = \tilde{f}'(0) = 0$ we deduce from (6.35) that

$$(6.37)\quad \|\mathcal{U}(u_0) - \mathcal{P}_\gamma u_0\|_{\mathbb{L}_{2\gamma}(\Phi_b)} \le C\|\mathcal{U}(u_0) - \mathcal{P}_\gamma u_0\|_{\mathbb{L}_{2\gamma}(W_b^{2-\mu,q})} + \\ + C\|\mathcal{U}(u_0)\|^2_{\mathbb{L}_\gamma(W_b^{2-\mu,q})} \le C_1\|u_0\|^2_{\mathbb{B}_{\sigma,\xi_0}}$$

Let us define now $\mathcal{U}_0(u_0) = \mathcal{U}(u_0)\big|_{t=0}$. Then (6.37) together with the definition of S_γ imply the estimate (6.32). The assertion $\mathcal{U}_0(B(0, \mu_0, \mathbb{B}_{\sigma,\xi_0})) \subset \mathcal{A}$ follows immediately from description (3.2) of the attractor $\mathcal{A}$ and from the fact that the solution $u(t) = \mathcal{U}(u_0)(t)$ of the problem (6.1) which is defined for the first only for $t \le 0$ can be extended due to Theorems 2.1 and 2.2 to a complete solution $u(t)$, $t \in \mathbb{R}$ and $u(0) = \mathcal{U}_0(u_0)$.

Thus, it remains to verify the estimates (6.33). Let $u_0^1, u_0^2 \in B(0, r, \mathbb{B}_{\sigma,\xi_0})$, $u^i(t) := \mathcal{U}(u_0^i)(t)$ be the corresponding backward solutions of (6.1), $v_0 := u_0^1 - u_0^2$ and $v(t) := u^1(t) - u^2(t)$. Then this function satisfies the equation

$$(6.38)\quad v + \mathbb{T}_{2\gamma}(\tilde{f}(u^1) - \tilde{f}(u^2)) - \mathcal{P}_\gamma v_0 = 0$$

Let us fix $N \in \mathbb{R}_+$, $x_0 \in \mathbb{R}^n$ and the corresponding weight function $\theta_{N,x_0}(x) = (1+|x-x_0|^{2N})^{-1/2}$. The equation (6.38) together with Lemma 6.1 and Corollary 1.4 imply that

$$\|v - \mathcal{P}_\gamma v_0\|_{\mathbb{L}_\gamma(W^{2-\mu,q}_{b,\theta_{N,x_0}})} \le C_N \|\tilde f(u^1) - \tilde f(u^2)\|_{\mathbb{L}_{2\gamma}(L^q_{b,\theta_{N,x_0}})} \tag{6.39}$$

where C_N is independent of x_0.

Recall that $\tilde f \in C^2$ and $\tilde f(0) = \tilde f'(0) = 0$, consequently

$$|\tilde f(u^1) - \tilde f(u^2)| \le Q(|u^1| + |u^2|)(|u^1| + |u^2|)|u^1 - u^2| \tag{6.40}$$

for a some monotonic function Q. The estimates (6.40) and (6.36) imply that

$$\begin{aligned} \|\tilde f(u^1) - \tilde f(u^2)\|_{\mathbb{L}_{2\gamma}(L^q_{b,\theta_{N,x_0}})} \le \widehat{Q}\left(\|u^1\|_{\mathbb{L}_0(W^{2-\mu,q}_b)} + \|u^2\|_{\mathbb{L}_0(W^{2-\mu,q}_b)}\right) \times \\ \times \left(\|u^1\|_{\mathbb{L}_\gamma(W^{2-\mu,q}_b)} + \|u^2\|_{\mathbb{L}_\gamma(W^{2-\mu,q}_b)}\right) \|v\|_{\mathbb{L}_\gamma(W^{2-\mu,q}_{b,\theta_{N,x_0}})} \le \\ \le \widehat{Q}(2Cr)2Cr\|v\|_{\mathbb{L}_\gamma(W^{2-\mu,q}_{b,\theta_{N,x_0}})} \end{aligned} \tag{6.41}$$

for every $B(0,r,\mathbb{B}_{\sigma,\xi_0})$. Decreasing r if necessary we may assume that

$$\begin{aligned} \|\tilde f(u^1) - \tilde f(u^2)\|_{\mathbb{L}_{2\gamma}(L^q_{b,\theta_{N,x_0}})} \le \\ \le \delta/C_N \left(\|v - \mathcal{P}_\gamma v_0\|_{\mathbb{L}_\gamma(W^{2-\mu,q}_{b,\theta_{N,x_0}})} + \|\mathcal{P}_\gamma v_0\|_{\mathbb{L}_\gamma(W^{2-\mu,q}_{b,\theta_{N,x_0}})}\right) \end{aligned} \tag{6.42}$$

where $\delta = \delta(r)$ can be fixed arbitrarily small (if $r > 0$ is small enough). The estimates (6.39) and (6.42) yield that

$$\|v - \mathcal{P}_\gamma v_0\|_{\mathbb{L}_\gamma(W^{2-\mu,q}_{b,\theta_{N,x_0}})} \le \delta \|\mathcal{P}_\gamma v_0\|_{\mathbb{L}_\gamma(W^{2-\mu,q}_{b,\theta_{N,x_0}})} \tag{6.43}$$

Applying (6.26) to the estimate (6.43) (and assuming that r is sufficiently small that $\delta < 1/2$) we derive that

$$\|v\|_{\mathbb{L}_\gamma(W^{2-\mu,q}_{b,\theta_{N,x_0}})} \le C_2 \|v_0\|_{\mathbb{L}^\infty_{b,\theta_{N/q,x_0}}} \tag{6.44}$$

Note that the function $v(t)$ is a solution of (2.57), consequently due to (6.26), (6.44) and due to the smoothing property (2.61)

$$\begin{aligned} \|\mathcal{U}_0(u_0^1) - \mathcal{U}_0(u_0^2)\|_{\Phi_{b,\theta_{N,x_0}}} \le \|\mathcal{U}(u_0^1) - \mathcal{U}(u_0^2)\|_{\mathbb{L}_\gamma(W^{2-\mu,q}_{b,\theta_{N,x_0}})} \le \\ \le C\|v\|_{\mathbb{L}_\gamma(W^{2-\mu,q}_{b,\theta_{N,x_0}})} \le C_1 \|v_0\|_{L^\infty_{b,\theta_{N/q,x_0}}} \end{aligned} \tag{6.45}$$

Since the constant C_1 in (6.45) is independent of x_0 then the first estimate of (6.33) is an immediate corollary of this estimate. Thus, it remains to prove only the second one. In order to do so we recall that $\Pi_e S_\gamma u_0 \equiv \operatorname{Re} u_0$ (see Lemma 6.2) and consequently (due to (6.30))

$$\begin{aligned} \|S_\gamma v_0\|_{W^{2-\mu,q}_{b,\theta_{N,x_0}}} \ge C\|S_\gamma v_0\|_{L^\infty_{b,\theta_{N/q,x_0}}} \ge \\ \ge C_1 \|\operatorname{Re} v_0\|_{L^\infty_{b,\theta_{N/q,x_0}}} \ge C_2 \|v_0\|_{L^\infty_{b,\theta_{N/q,x_0}}} \end{aligned} \tag{6.46}$$

The estimates (6.43) and (6.26) imply that

$$\|S_\gamma v_0\|_{W^{2-\mu,q}_{b,\theta_N,x_0}} \le \|v(0)\|_{\Phi_{b,\theta_N,x_0}} + C\delta\|v_0\|_{L^\infty_{b,\theta_{N/q},x_0}} \tag{6.47}$$

Combining (6.46) and (6.47) and fixing $\delta > 0$ in such a way that $C\delta < C_2/2$ we finally obtain that

$$\|v_0\|_{L^\infty_{b,\theta_{N/q},x_0}} \le C_3\|v(0)\|_{\Phi_{b,\theta_N,x_0}} \tag{6.48}$$

Theorem 6.1 is proved.

Corollary 6.4. *Let the assumptions of Theorem 6.1 be valid and let ϕ be a weight function with the polynomial rate of growth (see* (1.18)*). Then the map $\mathcal{U}_0$ realizes the Lipschitz continuous homeomorphism between $B(0,r,\mathbb{B}_{\sigma,\xi_0})$ and it's image $\mathcal{U}_0(B(0,r,\mathbb{B}_{\sigma,\xi_0}))$ in the following sense:*

$$C_1\|u_0^1 - u_0^2\|_{L^\infty_{b,\phi}} \le \|\mathcal{U}_0(u_0^1) - \mathcal{U}_0(u_0^2)\|_{\Phi_{b,\phi^q}} \le C_2\|u_0^1 - u_0^2\|_{L^\infty_{b,\phi}} \tag{6.49}$$

Indeed, the estimate (6.49) is an immediate corollary of (6.33) and (1.19).

Remark 6.1. Recall that the spaces $\mathbb{B}_\sigma$ and $\mathbb{B}_{\sigma,\xi_0}$ are isomorphic and the multiplication operator $\mathcal{G}_{\xi_0}u_0 := e^{i\xi_0\cdot x}u_0$ realizes this isomorphism. Moreover, since $|e^{i\xi_0\cdot x}| = 1$ then this isomorphism preserves the norms $\|\cdot, B^R_{x_0}\|_{0,\infty}$, particularly $\mathcal{G}_{\xi_0}B(0,r,\mathbb{B}_\sigma) = B(0,r,\mathbb{B}_{\sigma,\xi_0})$ and the operator

$$\tilde{\mathcal{U}}_0 := \mathcal{U}_0 \circ \mathcal{G}_{\xi_0} : B(0,r,\mathbb{B}_\sigma) \to \mathcal{A} \tag{6.50}$$

realizes a Lipschitz continuous embedding which satisfies the estimates (6.49)

Corollary 6.5. *Let $\{T_h, h \in \mathbb{R}^n\}$ be group of spatial shifts: $(T_hu)(x) := u(x+h)$ and let $\mathbb{K} := B(0,r,\mathbb{B}_\sigma(\mathbb{R}^k,\mathbb{C}))$, where r is the same as in Theorem 6.1. Then, evidently, $T_h\mathcal{A} = \mathcal{A}$ and $T_h\mathbb{K} = \mathbb{K}$. Moreover the map $\tilde{\mathcal{U}}_0 : \mathbb{K} \to \mathcal{A}$ commutes with this group:*

$$T_h\tilde{\mathcal{U}}_0(u_0) = \tilde{\mathcal{U}}_0(T_hu_0), \quad \textit{for every } h \in \mathbb{R}^n \tag{6.51}$$

Indeed, the assertion (6.51) is an immediate corollary of our construction of the map $\tilde{\mathcal{U}}_0$ and of the uniqueness part of the implicit function theorem.

Corollary 6.6. *Let $u_0^1, u_0^2 \in B(0,\mu,\mathbb{B}_{\sigma,\xi_0})$ and $\mu \le r$ (where r,σ,ξ_0 are the same as in Theorem 6.1). Then for every $R > R_0$*

$$\|\mathcal{U}_0(u_0^1) - \mathcal{U}_0(u_0^2)\|_{W^{2-\delta,p}_b(B^R_0)} \ge L\|\operatorname{Re}(u_0^1 - u_0^2)\|_{L^\infty(B^R_0)} - C\mu^2 \tag{6.52}$$

where C and L are independent of R.

Indeed,

$$\begin{aligned}
&\|\mathcal{U}_0(u_0^1) - \mathcal{U}_0(u_0^2)\|_{\Phi_b(B^R_0)} \ge \\
&\ge \|S_\gamma u_0^1 - S_\gamma u_0^2\|_{\Phi_b(B^R_0)} - \|\mathcal{U}_0(u_0^1) - S_\gamma u_0^1\|_{\Phi_b(\mathbb{R}^n)} + \|\mathcal{U}_0(u_0^2) - S_\gamma u_0^2\|_{\Phi_b(\mathbb{R}^n)} \ge \\
&\qquad \ge L\|S_\gamma u_0^1 - S_\gamma u_0^2\|_{L^\infty(B^R_0)} - C_1(\|u_0^1\|^2_{\mathbb{B}_{\sigma,\xi_0}} + \|u_0^2\|^2_{\mathbb{B}_{\sigma,\xi_0}}) \ge \\
&\qquad\qquad \ge L\|\operatorname{Re}(u_0^1 - u_0^2)\|_{L^\infty(B^R_0)} - 2C_1\mu^2
\end{aligned}$$

Here we have used the fact that $\Pi_e S_\gamma u_0 = \operatorname{Re} u_0$.

Now we are in a position to obtain the lower bounds for the ε-entropy of the attractor $\mathcal{A}$ of the equation (6.1).

Theorem 6.2. *Let the assumptions of Theorem 6.1 hold. Then the attractor $\mathcal{A}$ of the problem* (6.1) *possesses the following entropy estimates:*

$$(6.53)\qquad C_2R^n\ln\frac{1}{\varepsilon}\le\mathbb{H}_\varepsilon\left(\mathcal{A},W_b^{2,q}(B_0^R)\right)\le C_1(R+K\ln\frac{1}{\varepsilon})^n\ln\frac{1}{\varepsilon},\ \varepsilon\le\varepsilon_0<1$$

Moreover, for every $\delta>0$ there exists $C_\delta>0$ such that

$$(6.54)\qquad C_\delta\left(\ln\frac{1}{\varepsilon}\right)^{n+1-\delta}\le\mathbb{H}_\varepsilon\left(\mathcal{A},W_b^{2,q}(B_0^1)\right)\le C\left(\ln\frac{1}{\varepsilon}\right)^{n+1}$$

Proof. Indeed, let $\varepsilon>0$ be small enough, $\mu=\left(\frac{\varepsilon}{2CL}\right)^{1/2}\le r$ and functions $v_0^1,v_2^0\in B(0,\mu,\mathbb{B}^{Re}_{\sigma,\xi_0})$ be such that

$$(6.55)\qquad \|v_0^1-v_0^2\|_{L^\infty(B_0^R)}\ge\varepsilon/L$$

Then it follows from (6.52) that

$$(6.56)\qquad \|\mathcal{U}_0(\mathcal{R}v_0^1)-\mathcal{U}_0(\mathcal{R}v_0^2)\|_{W_b^{2,q}(B_0^R)}\ge\varepsilon/2$$

where $\mathcal{R}$ is the isomorphism constructed in Proposition 6.1.

The estimates (6.55),(6.56) together with the fact that $\mathcal{U}_0(\mathcal{R}v_0^i)\in\mathcal{A}$ imply that

$$(6.57)\qquad \mathbb{H}_{\varepsilon/4}\left(\mathcal{A},W_b^{2,q}(B_0^R)\right)\ge\mathbb{H}_{\varepsilon/L}\left(B(0,\left(\frac{\varepsilon}{2CL}\right)^{1/2},\mathbb{B}^{Re}_{\sigma,\xi_0}),C_b(B_0^R)\right)=$$
$$=\mathbb{H}_{(2C\varepsilon/L)^{1/2}}\left(B(0,1,\mathbb{B}^{Re}_{\sigma,\xi_0}),C_b(B_0^R)\right)$$

The estimates (6.53) and (6.54) are an immediate corollaries of (4.13) and (4.14) (see also Remark 4.2) and Theorem 5.1. Theorem 6.2 is proved.

Corollary 6.7. *Let the assumptions of Theorem 6.2 hold. Then*

$$(6.58)\qquad 0<C_1\ln\frac{1}{\varepsilon}\le\overline{\mathbb{H}}_\varepsilon(\mathcal{A})\le C_2\ln\frac{1}{\varepsilon}$$

and consequently

$$(6.59)\qquad 0<C_1\le\widehat{h}_{sp}(\mathcal{A})\le C_2<\infty$$

§7 The spatial complexity of the attractor and spatial chaos.

In this Section we continue to study the attractor of the spatially homogeneous system (6.1) in $\Omega=\mathbb{R}^n$ under the assumptions of Theorem 6.1. Recall that the group $\{T_h,h\in\mathbb{R}^n\}$ of spatial shifts acts on the attractor of (6.1)

$$(7.1)\qquad T_h\mathcal{A}=\mathcal{A},\quad (T_hu)(x):=u(x+h),\quad h\in\mathbb{R}^n$$

The main aim of this Section is to study the action of this group on the attractor from the dynamical point of view. Under this approach the semigroup (7.1) will be treated as a dynamical system with multidimensional 'time' $h\in\mathbb{R}^n$. (Note that in the particular case $n = 1$ we obtain a usual dynamical system with one-dimensional time.)

As a simple corollary of the estimates obtained in the previous Section (Theorem 6.2) we verify that the topological entropy $h_{sp}(\mathcal{A})$ of the semigroup (7.1) is infinite and define a new quantitative characteristic $\widehat{h}_{sp}(\mathcal{A})$ of the complexity of dynamics which is occurred to be finite and positive for the case of (7.1).

Recall that the usual way to indicate the chaotic behavior of a dynamical system $T_h : \mathcal{A} \to \mathcal{A}$ is to find a closed invariant subset $M \subset \mathcal{A}$ in the corresponding phase space and construct a homeomorphism $\tau : M \to \mathcal{M}$ such that

$$\tau : (T_h\big|_M, M) \to (\widehat{T}_h, \mathcal{M}), \quad , \ \widehat{T}_h := \tau \circ T_h \circ \tau^{-1} \tag{7.2}$$

where $(\widehat{T}_h, \mathcal{M})$ is a some model example of the dynamical system the chaotic behavior of which is evident. Note also that usually the homeomorphism (7.2) is constructed only for the appropriate *discrete* subgroup of T_h and the model examples $(\widehat{T}_h, \mathcal{M})$ are the appropriate Bernulli shifts (see e.g. [21]).

It is worth to emphasize that the (multidimensional) symbolic dynamics with finite number of symbols (Bernulli shifts) are not adequate in order to understood the spatial dynamics (7.1) because the topological entropy of such shifts is finite but in our situation we have the dynamics with the infinite topological entropy. That is why we introduce below a new model example of chaos $(\widehat{T}_h, \mathcal{M})$ which is close to the standard Bernulli shifts but adopted to the case of infinite topological entropy and construct the Lipschitz continuous embedding of this model to (7.1).

We start our exposition with the following definition.

Definition 7.1. Let $\phi(x) > 0$, $\phi \in C_b(\mathbb{R}^n)$ be a weight function which satisfies $\lim_{|x|\to\infty} \phi(x) = 0$ and let $\mathcal{A}$ be a compact set in $\Phi_{b,\phi}$ invariant with respect to T_h action. Then for every $R \in \mathbb{R}_+$ we define a new metric on $\mathcal{A}$ by formula

$$d_{R,\phi}(x, y) := \sup_{h\in[-R,R]^n} \|T_h x - T_h y\|_{\Phi_{b,\phi}}, \quad x, y \in \mathcal{A} \tag{7.3}$$

Define now the following characteristics:

$$h_{sp}(\mathcal{A}, \phi) = h_{sp}(\mathcal{A}, \phi, T_h) := \lim_{\varepsilon\to 0} \limsup_{R\to\infty} \frac{1}{(2R)^n} \mathbb{H}_\varepsilon(\mathcal{A}, d_{R,\phi}) \tag{7.4}$$

$$\widehat{h}_{sp}(\mathcal{A}, \phi) := \limsup_{\varepsilon\to 0} \frac{1}{\ln 1/\varepsilon} \limsup_{R\to\infty} \frac{1}{(2R)^n} \mathbb{H}_\varepsilon(\mathcal{A}, d_{R,\phi}) \tag{7.5}$$

Remark 7.1. The quantity (4.4) coincide with the definition of the topological entropy for the group $T_h : \mathcal{A} \to \mathcal{A}$ (adopted to the n-dimensional case) (see e.g. [21]) and (7.5) is one of possible generalizations of this concept for the case where the topological entropy is infinite. That is why we will call (7.5) as the modified topological entropy.

The following simple lemma is very important for our purposes.

Lemma 7.1. *Let the above assumptions hold. Then for every ϕ such as in Definition 7.1*

$$h_{sp}(\mathcal{A}, \phi) = h_{sp}(\mathcal{A}) := \lim_{\varepsilon\to 0} \limsup_{R\to\infty} \frac{1}{(2R)^n} \mathbb{H}_\varepsilon(\mathcal{A}, W^{2,q}_b([-R, R]^n)) \tag{7.6}$$

and analogously

$$\widehat{h}_{sp}(\mathcal{A},\phi) = h_{sp}(\mathcal{A}) := \lim_{\varepsilon\to 0}\frac{1}{\ln 1/\varepsilon}\limsup_{R\to\infty}\frac{1}{(2R)^n}\mathbb{H}_\varepsilon(\mathcal{A}, W_b^{2,q}([-R,R]^n)) \tag{7.7}$$

Particularly these quantitatives are independent of the choice of the weight ϕ.

Proof. Indeed, since $\phi(x)\to 0$ as $|x|\to\infty$ then for every $\varepsilon>0$ there is $L = L(\varepsilon)$ such that $\phi(x)<\varepsilon$ for $|x|>L(\varepsilon)$, consequently

$$\mathbb{H}_\varepsilon(\mathcal{A}, d_{R,\phi}) \le \mathbb{H}_{\varepsilon/C}(\mathcal{A}, W_b^{2,q}([-R-L(\varepsilon), R+L(\varepsilon)]^n) \tag{7.8}$$

for the appropriate C which is independent of R. Therefore

$$h_{sp}(\mathcal{A},\phi) \le h_{sp}(\mathcal{A}) \text{ and } \widehat{h}_{sp}(\mathcal{A},\phi)\le \widehat{h}_{sp}(\mathcal{A}) \tag{7.9}$$

The opposite inequalities follow from the evident estimate

$$\sup_{h\in[-R,R]^n}\phi(x+h)\ge\phi(0)>0, \quad \text{for } |x_i|\le R \tag{7.10}$$

Lemma 7.1 is proved.

Remark 7.2. It is well known (see e.g. [21]) that the topological entropy $h_{sp}(\mathcal{A})$ depends only on the topology on $\mathcal{A}$ and independent of the choice of the metric preserving the topology. Note, however, that the modified topological entropy $\widehat{h}_{sp}(\mathcal{A})$ does not possess this property and rigorously speaking is not a topological invariant.

Note also that $\widehat{h}_{sp}$ is evidently a Lipschitz invariant, i.e. preserves under the Lipschitz continuous homeomorphisms. Moreover, if τ is Holder continuous with the Holder constant $0<\alpha<1$ then

$$\widehat{h}_{sp}(\tau(M)) \le \frac{1}{\alpha}\widehat{h}_{sp}(M) \tag{7.11}$$

(compare with the fractal dimension).

The following theorem justifies our choice of generalization of the topological entropy.

Theorem 7.1. *Let the assumptions of Theorem 6.2 be valid and let $\mathcal{A}$ be the attractor of the equation* (6.1)*. Then the group $\{T_h, h\in\mathbb{R}^n\}$ of spatial shifts on the attractor has the infinite topological entropy*

$$h_{sp}(\mathcal{A}) = \infty \tag{7.12}$$

Moreover, the modified topological entropy of it is finite and strictly positive:

$$0 < C_1 \le \widehat{h}_{sp}(\mathcal{A}) \le C_2 < \infty \tag{7.13}$$

Indeed, the assertion of the theorem is an immediate corollary of Corollary 6.7 and Lemma 7.1.

Let us study now the spatial chaos generated the action of $\{T_h, h\in\mathbb{R}^n\}$ on the attractor $\mathcal{A}$. We give for the first the model construction (7.2) for the case of continuous dynamics ($h\in\mathbb{R}^n$) and after that we simplify this model for the case of discrete dynamics ($h\in\mathbb{Z}^n$).

Theorem 7.2. *Let the assumptions of Theorem 6.1 be valid and let r and σ be the same as in Theorem 6.1. Let also $\mathbb{K}$ be the ball $B(0,r,\mathbb{B}_\sigma)$ endowed by the local topology of $L^\infty_{loc}(\mathbb{R}^n)$. Then the map $\tilde{\mathcal{U}}_0 : \mathbb{K} \to \mathcal{A}$ defined in* (6.50) *realizes a homeomorphism*

$$\tilde{\mathcal{U}}_0 : (T_h, \mathbb{K}) \to (T_h, \tilde{\mathcal{U}}_0(\mathbb{K})) \tag{7.14}$$

Moreover, this homeomorphism is Lipschitz continuous if we endowed the spaces $\mathbb{K}$ and $\mathcal{A}$ by the topology $L^\infty_{b,\phi}$ and Φ_{b,ϕ^q} respectively (where ϕ is an arbitrary weight function with the polynomial rate of growth) and consequently this homeomorphism preserves the modified topological entropy:

$$0 < C_1 \le \widehat{h}_{sp}(\mathbb{K}) = \widehat{h}_{sp}(\tilde{\mathcal{U}}_0(\mathbb{K})) \le C_2 < \infty \tag{7.15}$$

Indeed, the assertion of this theorem is an immediate corollary of Theorem 6.1 and Corollaries 6.4 and 6.5.

Thus, the r-ball $\mathbb{K}$ of the space $\mathbb{B}_\sigma$ together with the group of spatial shifts $\{T_h, h \in \mathbb{R}^n\}$ acting on it can be considered as a model example for the topological description of the spatial chaos in the reaction-diffusion systems in unbounded domains. Note however that this model is rather complicated by itself and it seems reasonable to simplify it. To this end we restrict ourselves to consider only the action of a discrete subgroup $\{T_h, h \in \mathbb{Z}^n\}$ of the group of spatial shifts and use the Kotelnikov-Cartrait interpolation formula for representing the functions from $\mathbb{B}_\sigma$ (see e.g. [22], [37]).

Proposition 7.1. *Every function $u(x)$ from the class $\mathbb{B}_{\sigma'}$ can be represented in the following form:*

$$u(x) = \Sigma_{l\in\mathbb{Z}^n} u(\delta k) g_{\rho,k}(x), \quad \rho > 0 \tag{7.16}$$

where $\delta = \frac{\pi}{\sigma'+\rho}$ and

$$g_{\rho,k}(x) := \Pi_{j=1}^n \frac{\sin\rho(x^j - \delta k^j)\cdot \sin(\sigma'+\rho)(x^j - \delta k^j)}{\rho(\sigma'+\rho)(x^j-\delta k^j)^2} \tag{7.17}$$

Moreover, $g_{\rho,k} \in \mathbb{B}_{\sigma'+2\rho}$.

Let $\mathbb{D} := \{z \in \mathbb{C} : |z| \le 1\}$ be a unitary disk on the complex plane and let $\mathcal{M} := \mathbb{D}^{\mathbb{Z}^n}$ be the space of all functions $v : \mathbb{Z}^n \to \mathbb{D}$. We endow this space by a Frechet topology generated by the following system of seminorms:

$$\|v, B_0^R\|_{0,\infty} := \sup_{l\in\mathbb{Z}^n, |l|\le R} |v(l)| \tag{7.18}$$

and denote the space thus obtained by $\mathcal{M}_{loc}$ (It is evident that $\mathcal{M}_{loc}$ is a compact metric space and it's topology coincide with the Tikhonov's topology on the Descartes product $\mathbb{D}^{\mathbb{Z}^n}$). The spaces $\mathcal{M}_b$ and $\mathcal{M}_{b,\phi}$ where ϕ is a weight function can be defined analogously.

Fix now $\sigma', \rho > 0$ in such a way that $\sigma' + 2\rho < \sigma$ and define a map $\kappa : \mathcal{M} \to \mathbb{B}_\sigma$ by the expression

$$\kappa(v) := \sum_{l\in\mathbb{Z}^n} v(l) g_{\rho,l}(x) \tag{7.19}$$

where the functions $g_{\rho,l}$ are defined in (7.17). Then the following is true.

Lemma 7.2. *Let the above assumptions hold and let* $0 < \nu < 1$. *Then*

$$|\kappa(v)(x)| \le C \sup_{l\in\mathbb{Z}^n} |v(l)| \left(\Pi_{j=1}^n (1 + |x^j - l^j|^2)\right)^{-\nu/2} \tag{7.20}$$

Moreover, for every $R > \sqrt{n}$

$$\|\kappa(v), B_0^{\delta R}\|_{0,\infty} \ge \|v, B_0^R\|_{0,\infty} \tag{7.21}$$

Proof. Indeed, the estimate (7.21) follows immediately from the fact that $\kappa(v)(\delta l) = v(l)$ for every $l \in \mathbb{Z}$ (see (7.16) and (7.17)).

The proof the estimate (7.20) is based on the evident estimate

$$|g_{\rho,l}(x)| \le \frac{C}{\Pi_{j=1}^n (1 + |x^j - l^j|^2)}\ , \quad l \in \mathbb{Z}^n, \quad x \in \mathbb{R}^n \tag{7.22}$$

and also can be verified in a direct way.

Corollary 7.2. *Let the above assumptions hold. Then there is a constant* $C = C(\sigma', \rho)$ *such that*

$$\|\kappa(\mathcal{M})\|_{L_b^\infty(\mathbb{R}^n)} \le C \tag{7.23}$$

Moreover, for every weight function ϕ *with a polynomial rate of of growth* $\nu < 1$ *(see* (1.18)*) the following estimate is valid: Then*

$$C_1 \|v\|_{\mathcal{M}_{b,\phi}} \le \|\kappa(v)\|_{L_{b,\phi}^\infty} \le C_2 \|v\|_{\mathcal{M}_{b,\phi}} \tag{7.24}$$

The assertions of this corollary follow from the estimates (7.20), (7.21) and (1.19).

Let now $\sigma, r > 0$ be the same as in Theorem 6.1 and 7.2. Then the estimate (7.23) implies that the map

$$\tilde{\kappa}(v) := \frac{r}{C}\kappa(v), v \in \mathcal{M} \tag{7.25}$$

where C is defined in (7.23) realizes an embedding $\mathcal{M}$ to $\mathbb{K}$. Moreover, the estimate (7.24) remains valid for $\tilde{\kappa}$ as well and shows that this embedding is Lipschitz continuous in the appropriate metric.

Let us consider now a discrete subgroup $T_h' := \{T_h, h = \delta l, l \in \mathbb{N}^n\}$ of the semigroup of spatial shifts acting on $\mathbb{K}$ and on the attractor $\mathcal{A}$ of the equation (6.1). Define also the action of this subgroup on the space $\mathcal{M}$ by formula

$$(T_{\delta l}' v)(m) := v(m + l), \quad v \in \mathcal{M}, \quad l, m \in \mathbb{Z}^n \tag{7.26}$$

Then the following is true.

Lemma 7.3. *Let the above assumptions hold. Then the set* $\tilde{\kappa}(\mathcal{M})$ *is invariant with respect to the discrete group* T_h' *and this group commutes with the map* $\tilde{\kappa}$ *defined by* (7.25)*, i.e.*

$$\tilde{\kappa} \circ T_h' = T_h' \circ \tilde{\kappa} \tag{7.27}$$

Indeed, the assertion of the lemma is an immediate corollary of the fact that $\kappa(v)(\delta l) \equiv v(l)$.

Note now that the topological entropy h_{sp} and the modified topological entropy $\widehat{h}_{sp}$ can be defined analogously to Definition 7.1 for a discrete groups as well. Moreover, the assertions of Lemma 7.1 and Remark 7.2 also remains valid for this case. Consequently, (due to (7.24)) the map

$$\tilde{\kappa} : (T'_h, \mathcal{M}) \to (T'_h, \tilde{\kappa}(\mathcal{M})) \subset (T'_h, \mathbb{K}) \tag{7.28}$$

preserves the modified topological entropy

$$\widehat{h}_{sp}(T'_h, \mathcal{M}) = \widehat{h}_{sp}(T'_h, \tilde{\kappa}(\mathcal{M})) \tag{7.29}$$

Thus, for the case of discrete group of shifts T'_h, we have constructed the Lipschitz continuous embedding of the model dynamical system $(T'_h, \mathcal{M})$ to the dynamical system $(T'_h, \mathbb{K})$. (see (7.2)).

Combining this embedding with the embedding, constructed in Theorem 7.2 we obtain the following result.

Theorem 7.3. *Let the assumptions of Theorem 7.2 be valid and let T'_h be a discrete subgroup of spatial shifts, $h = \delta l$, $l \in \mathbb{Z}^n$. Then the map $\tau = \tilde{\mathcal{U}_0} \circ \tilde{\kappa}$ realizes a Lipschitz continuous (in weighted metrics described in Corollary 7.2) isomorphism between $\mathcal{M}$ and $\tau(\mathcal{M}) \subset \mathcal{A}$ which preserves the action of the group T'_h:*

$$\tau : (T'_h, \mathcal{M}) \to (T'_h, \tau(\mathcal{M})) \tag{7.30}$$

and consequently this homeomorphism preserves the modified topological entropy:

$$0 < \widehat{h}_{sp}(\mathcal{M}, T'_h) = \widehat{h}_{sp}(\tau(\mathcal{M}), T'_h) \tag{7.31}$$

Thus, we have constructed the Lipschitz continuous embedding of the model dynamical system $(T'_h, \mathcal{M})$ to the dynamical system $(T'_h, \mathcal{A})$, generated by the discrete spatial shifts on the attractor $\mathcal{A}$ of the equation (6.1). Note, that if we restrict ourselves to consider only the subset $\mathcal{M}_N \subset \mathcal{M}$ of functions $v : \mathbb{Z}^n \to \{a_1, \cdots a_N\}$ where $a_1, \cdots, a_N \in \mathbb{D}$ are arbitrary different complex numbers from the unitary ball, we obtain the standard symbolic dynamics with N symbols (multidimensional Bernulli shifts). Consequently Theorem 7.3 admits to embed the symbolic dynamics with N symbols into the discrete spatial shifts of the attractor $\mathcal{A}$ for every $N \in \mathbb{N}$. Moreover, the following theorem shows that an arbitrary *finite* dimensional (discrete) dynamics can be realized as a restriction of the discrete spatial shifts to the appropriate invariant subset of the attractor.

Theorem 7.4. *Let the assumptions of the previous theorem holds, let $K \subset \mathbb{C}^N$ be an arbitrary compact set in $\mathbb{C}^N$, and $\phi : K \to K$ be a homeomorphism. Define a dynamical system $\{G_n, n \in \mathbb{Z}\}$ on K by iteration of this homeomorphism*

$$G_n z := (\phi)^n z, \quad z \in K \tag{7.32}$$

Then there exists a homeomorphism $\tau : K \to \tau(K) \subset \mathcal{A}$ such that

$$\tau \circ G_n = T_{n\vec{p}} \circ \tau, \quad n \in \mathbb{Z} \tag{7.33}$$

where $\vec{p} := N\delta e_1 = N\delta(1,0,\cdots,0)$ and δ is the same as in Theorem 7.3.

Proof. Due to Theorem 7.3 it is sufficient to construct only the embedding of this system to a model one $(T'_h, \mathcal{M})$. Note also that without loss of generality we may assume that K is a subset of N-dimensional polydisc $K \subset \mathbb{D}^N$. Let us define an embedding $\theta : K \to \mathcal{M}$ by formula

$$\theta(z)(l_1, l_2, \cdots, l_n) := G_n(z)_k, \quad \text{where } l \in \mathbb{Z}^n, \\ l_1 = nN + k, \quad n \in \mathbb{Z}, \quad k \in \{0, 1, \cdots, N-1\}, \quad z \in K \subset \mathbb{D}^N \tag{7.34}$$

It is not difficult to verify that $\theta : K \to \theta(K) \subset \mathcal{M}$ is really a homeomorphism (since $G_n : K \to K$ are homeomorphisms). Moreover, it follows from the definition of θ that

$$\theta(G_n z) = T_{nNe_1}\theta(z), \quad z \in K, \quad n \in \mathbb{Z} \tag{7.35}$$

The assertion of the theorem is an immediate corollary of (7.35) and Theorem 7.3.

Remark 7.3. For simplicity we have formulated and proved the embedding theorem 7.4 only for the dynamical system (G_n, K) with one dimensional 'time' but it's generalization for the multidimensional case is straightforward.

§8 The temporal evolution of spatial chaos and the spatial complexity of individual trajectories

In the previous sections we construct a number of various invariant with respect to spatial shifts subsets $B \subset \mathcal{A}$ of the attractor the restrictions of $\{T_h, h \in \mathbb{R}^n\}$ to which demonstrate the chaotic behavior, have infinite topological entropy $h_{sp}(B) = \infty$, positive modified entropy $\widehat{h}_{sp}(B) > 0$ and so on. Note however that all sets thus constructed are not invariant with respect to the temporal dynamics $\{S_t, t \geq 0\}$ generated by the equation (6.1) (in a fact the image $\tilde{U}_0(\mathbb{K})$ constructed in Theorem 7.2 belongs to an exponentially unstable manifold of zero equilibria point). Thus, it seems reasonable to study the spatial complexity of sets S_tB, $t \geq 0$, where B is a spatially invariant subset of the attractor $\mathcal{A}$.

We start with a trivial corollary of the estimates formulated in Theorem 2.3.

Lemma 8.1. *Let the assumptions of Theorem 6.2 hold and let B be a compact in Φ_{loc} invariant with respect to the spatial shifts $\{T_h, h \in \mathbb{R}^n\}$ subset of the phase space Φ_b of the equation* (6.1). *Then*

$$h_{sp}(S_tB) \leq h_{sp}(B), \quad \widehat{h}_{sp}(S_tB) \leq \widehat{h}_{sp}(B), \quad t \geq 0 \tag{8.1}$$

where $S_t : \Phi_b \to \Phi_b$ is a semigroup, generated by the equation (6.1).

Proof. Indeed, the set B is evidently bounded in Φ_b and consequently due to the estimate (2.61) and (1.3) the semigroup S_t is Lipschitz continuous in the space $\Phi_{b,\phi}$ for every weight function which satisfies the assumption (1.1). But the (modified) topological entropy does not increase under the Lipschitz continuous mappings (see Remark 7.2). Lemma 8.1 is proved.

The main result of this Section is the following theorem.

Theorem 8.1. *Let the assumptions of Theorem 6.1 hold and let in addition the matrix a in the equation* (6.1) *is normal, i.e.*

$$aa^* = a^*a \tag{8.2}$$

Let B be a compact (in Φ_{loc}) invariant with respect to $\{T_h, h \in \mathbb{R}^n\}$ subset of the attractor $\mathcal{A}$. Then the quantitatives $h_{sp}(B)$ and $\widehat{h}_{sp}(B)$ preserves under the temporal dynamics:

$$h_{sp}(S_t B) = h_{sp}(B) \quad \text{and} \quad \widehat{h}_{sp}(S_t B) = \widehat{h}_{sp}(B), \quad t \ge 0 \tag{8.3}$$

Proof. The assertion of the theorem is a corollary of the following Lemma which claims that the semigroup S_t is backward Holder continuous on the attractor with the Holder exponent arbitrary close to 1.

Lemma 8.2. *Let the above assertions hold and let $u_1(t), u_2(t) \in \mathcal{A}$, $t \in \mathbb{R}$ be two arbitrary solutions of* (6.1) *belonging to the attractor. Then for every $0 < \alpha < 1$ and every fixed $T > 0$ there is $\varepsilon > 0$ and a constant $C = C(\alpha, T, \varepsilon)$ such that*

$$\|u_1(0) - u_2(0), B^1_{x_0}\|_{2,q} \le C \sup_{x\in\Omega} e^{-\varepsilon|x-x_0|} \|u_1(T) - u_2(T), B^1_x\|^{\alpha}_{0,2} \tag{8.4}$$

The proof of this Lemma is based on the following convexity result, formulated and proved in [2].

Proposition 8.1 [2]. *Let H be a Hilbert space and $B : D(B) \to H$ be a linear unbounded operator in it. Let also $v \in C^1([t_0, t_1], H) \cap C([t_0, t_1], D(B))$ be a solution of the following equation:*

$$\partial_t v - Bv = P(t)v, \quad \|P(t)\|_{H\to H} \le P_0 \tag{8.5}$$

Assume also that $B = B_+ + B'_- + B''_-$, where B_+ is a symmetric operator and B'_- and B''_- are skew symmetric operators such that for every $w \in H$

$$(B_+w, B'_-w)_H \ge -\gamma\|B_+w\|_H\|w\|_H - \beta\|w\|^2_H, \tag{8.6}$$

$$\|B''_-w\|^2_H \le \gamma\|B_+w\|_H\|w\|_H + \beta\|w\|^2_H \tag{8.7}$$

are satisfied. Let us define a new function

$$l(t) := 2\ln\|u(t)\|_H - \int_{t_0}^{t} \psi(s)\,ds, \quad \psi(t) := 2\frac{(P(t)u(t), u(t))}{\|u(t)\|^2_H} \tag{8.8}$$

Then the following inequality holds for every $t_0 \le t \le t_1$

$$l(t) \le \alpha_\pm l(t_0) + (1-\alpha_\pm)l(t_1) + e^{4\gamma(t_1-t_0)}(t_1-t_0)^2(8\gamma^2 + 4\beta + 2P_0^2) \tag{8.9}$$

where

$$\alpha_\pm := \frac{e^{\pm4\gamma t_1} - e^{\pm4\gamma t}}{e^{\pm4\gamma t_1} - e^{\pm4\gamma t_0}} \tag{8.10}$$

in (8.10) *one takes the negative sign if $l(t_0) \le l(t_1)$ and the positive sign if $l(t_0) \ge l(t_1)$.*

Corollary 8.1. *Let the assumptions of Lemma 8.1 hold and let it be known in addition that the solution $v(t)$ is defined on $(-\infty, t_1]$ and remain bounded: $\|v(t)\|_H \le K$. Then for every $\mu > 0$ and $t \in (-\infty, t_1)$ there is a constant $C = C(t, t_1, \mu, K)$ such that*

$$\|u(t)\|_H \le C\|u(t_1)\|_H^{\alpha}, \quad \alpha := e^{4\gamma(t-t_1)} - \mu \tag{8.11}$$

Proof. Indeed, applying the exponent to the both sides of the inequality (8.9) and taking into the account that $-2P_0(t-t_0) \le \int_{t_0}^t \psi(s)\,ds \le 2P_0(t-t_0)$ we derive that

$$\|u(t)\|_H \le C(t, t_1, t_0)\|u(t_1)\|_H^{1-\alpha_\pm}\|u(t_2)\|_H^{\alpha_\pm} \tag{8.12}$$

Since $\|u(t_2)\|_H \le K$ then (8.12) implies the estimate

$$\|u(t)\|_H \le C'(K, t, t_0, t_1)\|u(t_1)\|_H^{\alpha} \tag{8.13}$$

where $\alpha = \min\{1-\alpha_+, 1-\alpha_+\}$. Let us fix now $t_2 = -N$ where $N > 0$ is large enough. Then

$$\alpha = 1 - \alpha_+ = \frac{e^{4\gamma t} - e^{-4\gamma N}}{e^{4\gamma t_1} - e^{-4\gamma N}} \to e^{-4\gamma(t_1-t)} \tag{8.14}$$

when $N \to \infty$. Therefore, for every $\mu > 0$ one can find $N = N(\mu)$, such that $\alpha \ge e^{-4\gamma(t_1-t)} - \mu$. Corollary 8.1 is proved.

Let us prove Lemma 8.1 now. Indeed, let $v(t) := u_1(t) - u_2(t)$ then this function evidently satisfies the equation

$$\partial_t v = a\Delta_x v - \lambda_0 v - l(t)v \tag{8.15}$$

where $l(t) := \int_0^1 f'(su_1(t) + (1-s)u_2(t))\,ds$. Recall, $u_i(t)$ are complete bounded solutions belonging to the attractor $\mathcal{A}$, consequently due to Theorems 2.1 and 3.1 $\|u_i(t)\|_{C_b(\mathbb{R}^n)} \le \|u_i\|_{\Phi_b} \le C$ and therefore the function $l(t)$ is uniformly bounded: $\|l(t)\|_{C_b(\mathbb{R}^n)} \le C_1$ and C_1 is independent of u_i.

Fix now an arbitrary $x_0 \in \mathbb{R}^n$ and consider a function $w_{x_0}(t) := v(t)\tilde\phi_{\varepsilon,x_0}$ where the weight function $\tilde\phi_{\varepsilon,x_0}$ is the same as in the proof of Lemma 6.1 and ε is a small parameter. Then it is not difficult to verify that this function satisfies the equation

$$\partial_t w_{x_0}(t) - a\Delta_x w_{x_0}(t) + K_1(x)w_{x_0}(t) + K_2(x)\nabla_x w_{x_0}(t) + l(t)w_{x_0}(t) = 0 \tag{8.16}$$

where

$$K_1(x)w := \left(\frac{\Delta_x\tilde\phi_{\varepsilon,x_0}}{\tilde\phi_{\varepsilon,x_0}} - 2\frac{|\nabla_x\tilde\phi_{\varepsilon,x_0}|^2}{\tilde\phi_{\varepsilon,x_0}^2}\right)aw \tag{8.17}$$

(8.18)

$$K_2(x)\nabla_x w = 2\tilde\phi_{\varepsilon,x_0}^{-1}\nabla_x\tilde\phi_{\varepsilon,x_0}.a\nabla_x w := 2\tilde\phi_{\varepsilon,x_0}^{-1}\sum_{i=1}^n \partial_{x_i}\tilde\phi_{\varepsilon,x_0}a\partial_{x_i}w$$

Moreover, it follows from (6.12) that

$$|K_i(x)| + |\nabla_x K_i(x)| \le C\varepsilon$$

for the appropriate constant C.

Let us verify now that the equation (8.16) satisfies all assumptions of Lemma 8.1. Indeed, let $H := [L^2(\mathbb{R}^n)]^k$, $Rw := K_2(x)\nabla_x w$,

$$B_+ = 1/2(a+a^*)\Delta_x - \lambda_0 - 1/2(R+R^*),\ \ B'_- := 1/2(a-a^*)\Delta_x,\ \ B''_- := -1/2(R-R^*)$$

and $P(t)w := -K_1(x)w - l(t)w$. Then evidently B_+ is symmetric and B'_- and B''_- are skew symmetric. In order to verify the assumptions (8.6) and (8.7) we compute firstly the operator R^*:

$$R^*w := -2\tilde\phi^{-1}_{\varepsilon,x_0}\nabla_x\tilde\phi_{\varepsilon,x_0}.a^*\nabla_x w - 2\nabla_x\cdot\left(\nabla_x\tilde\phi_{\varepsilon,x_0}\phi^{-1}_{\varepsilon,x_0}\right)a^*w \tag{8.19}$$

and consequently

$$\begin{aligned}(B_+w, B'_-w) = 1/4\,&((a+a^*)\Delta_x w, (a-a^*)\Delta_x w) - \\ &-\left(\tilde\phi^{-1}_{\varepsilon,x_0}\nabla_x\tilde\phi_{\varepsilon,x_0}.(a-a^*)\nabla_x w, (a-a^*)\Delta_x w\right) + \\ &+\left(\nabla_x\left(\nabla_x\tilde\phi_{\varepsilon,x_0}\tilde\phi^{-1}_{\varepsilon,x_0}\right)a^*w, (a-a^*)\Delta_x w\right)\end{aligned} \tag{8.20}$$

Since a is normal (see the assumption (8.2)) then the first term in the right-hand side of (8.21) is equal to zero identically. Integrating by parts in the second term we derive that

$$\begin{aligned}&\left(\tilde\phi^{-1}_{\varepsilon,x_0}\nabla_x\tilde\phi_{\varepsilon,x_0}.(a-a^*)\nabla_x w, (a-a^*)\Delta_x w\right) = \\ &= -1/2\left(\nabla_x(\tilde\phi^{-1}_{\varepsilon,x_0}\nabla_x\tilde\phi_{\varepsilon,x_0})(a-a^*)\nabla_x w, (a-a^*)\nabla_x w\right) \le C\varepsilon\|\nabla_x w\|^2_H\end{aligned} \tag{8.21}$$

It follows from the interpolation inequality, the regularity theorem for the Laplace operator in $\mathbb{R}^n$ and from the fact that $\varepsilon > 0$ is small enough that

$$\|\nabla_x w\|^2_H \le C\|w\|_{W^{2,2}(\mathbb{R}^n)}\|w\|_H \le C_1\|B_+w\|_H\|w\|_H \tag{8.22}$$

And finally due to the Holder inequality

$$\begin{aligned}&\left(\nabla_x\left(\nabla_x\tilde\phi_{\varepsilon,x_0}\tilde\phi^{-1}_{\varepsilon,x_0}\right)a^*w, (a-a^*)\Delta_x w\right) \ge \\ &\qquad\ge -C\varepsilon\|w\|_H\|\Delta_x w\|_H \ge -C_2\varepsilon\|B_+w\|_H\|w\|_H\end{aligned} \tag{8.23}$$

Combining the estimates (8.20)–(8.23) we derive that

$$(B_+w, B'_-w) \ge -\gamma\|B_+w\|_H\|w\|_H,\quad \gamma = C\varepsilon$$

Thus, the assumption (8.6) is verified. Let us verify the assumption (8.7). Indeed, since B''_- is a first order differential operator then due to (8.22)

$$\|B''_-w\|^2_H \le C\varepsilon\left(\|\nabla_x w\|^2_H + \|w\|^2_H\right) \le C_1\varepsilon(\|B_+w\|_H\|w\|_H + \|w\|^2_H) \tag{8.24}$$

Thus, the assumption (8.7) is also verified.

Note also that $u_i(t) \in \mathcal{A}$ implies that $\|w_{x_0}(t)\|_{L^2(\mathbb{R}^n)} \le K$ where K is independent of x_0. Thus, all assumptions of Lemma 8.2 and Corollary 8.1 are verified and consequently according to (8.11) with $t_1 = T$ and $t = -1$

$$\|w_{x_0}(-1)\|_{0,2} \le C(\varepsilon, \mu, T)\|w_{x_0}(T)\|_{0,2}^{\alpha} \tag{8.25}$$

here $\alpha := e^{-C\varepsilon(T+1)} - \mu$, where $\mu > 0$ can be chosen arbitrarily small and the constant C is independent of x_0.

Note also that since ε, μ can be chosen arbitrarily small then the Holder exponent $\alpha < 1$ in (8.25) is arbitrarily close to 1.

The estimate (8.25) immediately implies that

$$\|v(-1), B^1_{x_0}\|_{0,2} \le C'(\alpha, \varepsilon, T) \sup_{x \in \mathbb{R}^n} \|v(T), B^1_x\|_{0,2}^{\alpha} \tag{8.26}$$

where α is arbitrarily close to 1 and $\varepsilon = \varepsilon(\alpha) > 0$. The estimate (8.4) is an immediate corollary of (8.26) and of the smoothing property (2.61). Lemma 8.2 is proved.

We are in a position now to complete the proof of Theorem 8.1. To this end we note that the estimate (8.4) implies that the restriction of $S_T\big|_{\mathcal{A}}$ on the attractor $\mathcal{A}$ is invertible and for every weight function ϕ with a polynomial rate of growth and for every $0 < \alpha < 1$ the operator $S_T^{-1}\big|_{\mathcal{A}}$ is uniformly Holder continuous with the exponent α

$$S_T^{-1} : \mathcal{A} \cap \Phi_{b,\phi} \to \mathcal{A} \cap \Phi_{b,\phi^\alpha} \tag{8.27}$$

i.e. for every $u_1, u_2 \in \mathcal{A}$

$$\|u_1 - u_2\|_{\Phi_{b,\phi^\alpha}} \le C(T, \alpha)\|S_T u_1 - S_T u_2\|^{\alpha}_{\Phi_{b,\phi}} \tag{8.28}$$

and consequently (due to Lemma 7.1 and the estimate (7.11))

$$\widehat{h}_{sp}(B) \le \alpha \widehat{h}_{sp}(S_T B), \text{ and } h_{sp}(B) = h_{sp}(S_T B) \tag{8.29}$$

Passing to the limit $\alpha \to 1$ in (8.29) and taking into the account the result of Lemma 8.1 we derive (8.3). Theorem 8.1 is proved.

Remark 8.1. Recall that we construct in Section 7 the set $B = \tilde{\mathcal{U}}_0(\mathbb{K}) \subset \mathcal{A}$ the restriction of spatial shifts on which is isomorphic to the model dynamics $(T_h, \mathbb{K})$ (or to $(T'_h, \mathcal{M})$ for discrete spatial shifts). The estimate (8.28) implies now that the set $S_T B \subset \mathcal{A}$ is also homeomorphic to $(T_h, \mathbb{K})$ (or $(T'_h, \mathcal{M})$ respectively). Thus, the spatial chaos constructed in Section 7 preserves under the time evolution $\{S_t, t \in \mathbb{R}_+\}$.

Let us study now the spatial complexity of individual solutions $u(t) \in \mathcal{A}$ of the equation (6.1). To this end we need the following definition.

Definition 8.1. Let $u_0 \in \mathcal{A}$. Denote by $\mathcal{H}_{sp}(u_0)$ the hull of this point with respect to the spatial shifts:

$$\mathcal{H}_{sp}(u_0) := \left[T_h u_0, h \in \mathbb{R}^n\right]_{\Phi_{loc}} \tag{8.30}$$

where $[\cdot]_{\Phi_{loc}}$ means a closure in the space Φ_{loc}, and define the quantitatives $h_{sp}(u_0)$ and $\widehat{h}_{sp}(u_0)$ by the following expressions:

$$h_{sp}(u_0) := h_{sp}(\mathcal{H}_{sp}(u_0)), \quad \widehat{h}_{sp}(u_0) := \widehat{h}_{sp}(\mathcal{H}_{sp}(u_0)) \tag{8.31}$$

(see Definition 7.1).

The following Corollary shows that the quantitatives (8.31) are constants along the trajectories of (6.1).

Corollary 8.2. *Let the assumptions of Theorem 8.1 be valid. Then for every $u_0 \in \mathcal{A}$ the following is true:*

$$h_{sp}(S_t u_0) = h_{sp}(u_0), \quad \widehat{h}_{sp}(S_t u_0) = \widehat{h}_{sp}(u_0), \quad t \geq 0 \tag{8.32}$$

Moreover, the quantity $\widehat{h}_{sp}(u_0)$ is finite for every $u_0 \in \mathcal{A}$ and there is a point $u_0 \in \mathcal{A}$ such that

$$h_{sp}(u_0) = \infty, \quad \widehat{h}_{sp}(u_0) > C > 0 \tag{8.33}$$

Proof. Indeed, the assertions (8.32) are immediate corollaries of Theorem 8.1. Thus, it remains only to verify the existence of a point u_0 which satisfies (8.33). To this end we recall that due to Theorem 7.3 it is sufficient to find a point $v_0 \in \mathcal{M}$ such that it's hull (with respect to discrete shifts group $\{T'_h, h \in \mathbb{Z}^n\}$ has a positive modified topological entropy. But it is not difficult to verify that the space $\mathcal{M}$ possesses a topologically transitive orbit, i.e., there exists $v_0 \in \mathcal{M}$ such that

$$\mathcal{M} = \left[T'_h v_0, h \in \mathbb{Z}^n\right]_{\mathcal{M}_{loc}} \tag{8.34}$$

Fixing now $u_0 := \tau(v_0) \subset \mathcal{A}$, where $\tau : \mathcal{M} \to \mathcal{A}$ is defined in Theorem 7.3 we obtain a point of $\mathcal{A}$ which satisfies (8.33). Corollary 8.3 is proved.

Remark 8.2. It follows from the proof of Corollary 8.2 that there is a point $u_0 \in \mathcal{A}$ with an extremely complicated spatial structure. Particularly $(T'_h, \mathcal{M}) \subset (T'_h, \mathcal{H}_{sp}(u_0))$ and consequently due to Theorem 7.4 any finite dimensional dynamics can be realized by restricting the discrete spatial shifts group to the appropriate subset of the hull $\mathcal{H}_{sp}(u_0)$ of this point.

In conclusion of the paper we illustrate the obtained results on the particular case of Ginzburg-Landau equation.

Example 8.1. Consider the equation

$$\partial_t u = (1 + i\alpha)\Delta_x u + Ru - (1 + i\beta)u|u|^{2\sigma}, \quad x \in \mathbb{R}^n \tag{8.35}$$

where $u = u(t,x) = u_1(t,x) + iu_2(t,x)$ is a complex valued unknown function $\alpha, \beta \in \mathbb{R}$, $R > 0$ and $\sigma > 0$ (see [25] and references therein).

It is not difficult to verify that our monotonicity assumption $f'(u) \geq -C$ is satisfied if

$$|\beta| \leq \frac{\sqrt{2\sigma + 1}}{\sigma} \tag{8.36}$$

the rest of the assumptions of (2.2) are satisfied for every α, β and $\sigma > 1/2$. The growth restriction (2.3) is valid for every σ if $n \leq 4$ and for $\sigma < 2/(n-4)$ if $n > 4$.

Thus, for $n \leq 4$ Theorems 2.1, 2.2 and 3.1 give the existence of solutions for (8.35), their L^∞-bounds and the attractor's existence if (8.36) is satisfied (α is arbitrary and $\sigma > 1/2$).

Note that zero equilibria point $u \equiv 0$ of the equation (8.35) is evidently exponentially unstable if $R > 0$. Thus, the assumptions of Theorem 6.2 is also satisfied (if $n \leq 4$, (8.36) is valid, $\sigma > 1/2$ and α is arbitrary) and consequently the entropy of the corresponding attractor possesses the upper and lower bounds (6.53) and (6.54).

Note also that the assumption (8.2) is also evidently satisfied for the equation (8.35) (written as a system with respect to real valued unknown variables $u = (u_1, u_2)$). Consequently, the results of Sections 7 and 8 are also valid under the above assumptions.

Remark 8.3. Note that we need the assumptions (2.2) and (2.3) in a fact only in order to establish the L^∞-bounds of solutions. If these bounds are known from somewhere then the results of Sections 3–8 remains valid without the restrictions (2.2) and (2.3). Particularly, the L^∞-estimates for the complex Ginzburg-Landau equation (8.35) under different assumptions on σ, α, β and n can be found in [19], [26], [27]. Consequently, the results of the paper remains valid for (8.35) under that assumptions as well.

8 July, 2000

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